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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... LAGRANGE

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On the Stability of Viscous Fluid Motions

James Serrin

Communicated by C. Truesdell

This paper deals with one aspect of the classical problem of hydrodynamical stability, namely, the determination of sufficient conditions for a basic (usually laminar) flow of an incompressible fluid to be stable under arbitrary disturbances. The technique to be applied is the well known method of energy, originated by Osbourne Reynolds and William McF. Orr and used since that time by many other writers[1]. In spite of this intensive study, it appears that a number of new results can be secured from the method, and it is to these that the paper is devoted.

Our main conclusion is a Reynolds number criterion for the stability of an *arbitrary* fluid motion in a bounded region. In particular, we show that a basic flow in a bounded region $\mathscr{V}$ is stable whenever its Reynolds number $\mathrm{Re} = V d/\nu$ is less than 5.71; here V is the maximum speed of the basic flow, d is the diameter of $\mathscr{V}$, and ν is the kinematic viscosity of the fluid. The number 5.71 in itself is neither especially good nor especially bad as a criterion for stability, but what *is* interesting is the fact that it is absolutely rigorous and applies independently of the geometry of the flow region and the particular flow involved. For this reason the result may appropriately be called a Reynolds number for universal stability. We also obtain similar criteria for the stability of flows in unbounded regions, and applications are made to the problem of uniqueness of steady flow.

In the second part of the paper (§ 4) we state a general variational problem connected with the stability of an arbitrary motion. The Euler-Lagrange equations corresponding to this problem bear an interesting and remarkable resemblance to the Navier-Stokes equations, but they are in general too difficult to solve, except for special cases.

The last part of the paper treats a particular example, the laminar Couette flow between rotating coaxial cylinders. The methods of the earlier sections of the paper, when applied to this case, yield the stability criterion

$$\left|\frac{\Omega_2-\Omega_1}{\nu}\right| \leqq (R_2^2 - R_1^2)\left\{\frac{\pi}{R_1 R_2 \log(R_2/R_1)}\right\}^2, \tag{1}$$

in which the notation is practically self-explanatory. The author knows of no other formula of this type which applies to *arbitrary* disturbances of the basic laminar motion. (The criteria of G. I. Taylor and others (*cf.* [*14*], Chapter 2)

[1] *Cf.* references [*1*]—[*9*] at the end of the paper.

all refer to infinitesimal disturbances of a special type; thus, although they have indisputable importance if one is considering the breakdown of already established flow, and give quite sharp results in certain limited situations, they do not by any means apply to the whole realm of possible disturbances of the motion.) Finally, the stability of Couette motion is considered from the point of view of the variational method noted above. The results here do not have the finality of criterion (1), but they are nevertheless of interest.

The paper begins with a derivation of a fundamental identity for the rate of change of energy of a perturbation motion. This formula is the basis for all that follows.

1. The Reynolds-Orr energy equation

We consider a basic fluid motion occupying a region $\mathscr{V}=\mathscr{V}(t)$ of space and subject to a prescribed velocity distribution on the boundary $\mathscr{S}$ of $\mathscr{V}$. In the cases of greatest interest $\mathscr{V}$ is bounded by material walls and the boundary conditions arise from the motion of these walls, as, for example, in the case of Couette flow. Now suppose the velocity field of the basic flow is altered at some initial instant $t=0$; it is natural to ask whether the subsequent motion, subject to the same boundary conditions, will alter only slightly from what it was, or whether it will change radically in character. To investigate this problem, we shall consider the energy $\mathscr{K}$ of the perturbation (difference) motion. If $\mathscr{K}$ tends to zero as $t\to\infty$, then the basic motion is said to be stable, or, more precisely, stable in the mean.

To be specific, suppose the region $\mathscr{V}$ is bounded, and let $\boldsymbol{v}$ and $\boldsymbol{v}'$ denote, respectively, the velocity vectors of the basic and altered motions. The velocity $\boldsymbol{u}=\boldsymbol{v}'-\boldsymbol{v}$ of the perturbation motion obviously satisfies

$$\boldsymbol{u}=0 \qquad \text{on } \mathscr{S}, \tag{2}$$

and its kinetic energy is given by

$$\mathscr{K}=\tfrac{1}{2}\int u^2 \tag{3}$$

(in writing integrals, we shall consistently omit the conventional volume infinitesimal; moreover, all integrals are understood to be extended over the entire region $\mathscr{V}$). The rate of change of $\mathscr{K}$ is governed by the fundamental formula

$$\boxed{\frac{d\mathscr{K}}{dt}=-\int(\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u}+\nu\,\operatorname{grad}\boldsymbol{u}:\operatorname{grad}\boldsymbol{u})} \tag{4}$$

essentially due to REYNOLDS and ORR. In this equation $\operatorname{grad}\boldsymbol{u}$ denotes the matrix with components $(\operatorname{grad}\boldsymbol{u})_{ik}=u_{k,i}$ and $\boldsymbol{D}$ denotes the deformation matrix of the basic motion, $D_{ik}=\frac{1}{2}(v_{i,k}+v_{k,i})$.

To prove (4), we begin with the observation that both $\boldsymbol{v}$ and $\boldsymbol{v}'$ satisfy the Navier-Stokes equation

$$\varrho\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\operatorname{grad}\boldsymbol{v}\right)=\varrho\boldsymbol{f}-\operatorname{grad}p+\mu\nabla^2\boldsymbol{v}.$$

By subtraction there arises

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\operatorname{grad}\boldsymbol{v}+\boldsymbol{v}'\cdot\operatorname{grad}\boldsymbol{u}=\operatorname{grad}\frac{p-p'}{\varrho}+\nu\nabla^2\boldsymbol{u}.$$

Forming the scalar product of this equation with the vector $\boldsymbol{u}$, and using the incompressibility conditions $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{v}' = 0$, then leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2} u^2\right) = -\left(\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} + \nu \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}\right) + \operatorname{div} \boldsymbol{\Phi}, \tag{5}$$

where

$$\boldsymbol{\Phi} = \frac{p-p'}{\varrho} \boldsymbol{u} + \nu \operatorname{grad}\left(\frac{1}{2} u^2\right) - \frac{1}{2} u^2 \boldsymbol{v}'.$$

On the other hand, we have obviously

$$\frac{d\mathscr{K}}{dt} = \int \frac{\partial}{\partial t}\left(\frac{1}{2} u^2\right) + \oint \frac{1}{2} u^2 \boldsymbol{v} \cdot \boldsymbol{n}, \tag{6}$$

the latter integral being taken over the entire boundary of $\mathscr{V}$. The required formula (4) now follows easily from (5), (6), the divergence theorem, and the fact that $\boldsymbol{u} = \boldsymbol{\Phi} = 0$ on $\mathscr{S}$.[2]

This derivation does not hold when the region $\mathscr{V}$ is unbounded, since both formula (6) and the divergence theorem are applicable only to bounded regions. Under suitable conditions on the asymptotic behavior of $\boldsymbol{v}, \boldsymbol{v}'$, however, the above formal steps can still be justified (we shall omit the details). Another justification of (4) for infinite regions is available whenever the flow geometry is such that the disturbances can be assumed spatially periodic at each instant. This will be the case, in particular, for the important Poiseuille and Couette flows. The disturbance $\boldsymbol{u}$ being supposed periodic in the direction of the axis of symmetry (what Lin calls sustained oscillations), the region $\mathscr{V}$ can then be chosen to cover exactly one period. Then the boundary integrals at either end of $\mathscr{V}$, neither of which vanishes separately, just cancel one another. Formula (4) may therefore be assumed to hold in these two important situations.

2. Criteria for universal stability

In this section we shall use the method of energy to establish certain criteria for the stability of arbitrary fluid motions. This method is based on the observation that if $\mathscr{K}$ tends to zero, then $\boldsymbol{u}$ must likewise tend to zero almost everywhere. Thus a basic flow will be stable (stable in the mean) provided the energy of any disturbance tends to zero as t increases. To apply the method one considers the right-hand side of (4): if it is negative for arbitrary non-vanishing vectors $\boldsymbol{u}$ satisfying $\operatorname{div} \boldsymbol{u} = 0$, then obviously $d\mathscr{K}/dt < 0$ and there is stability. Since the second term on the right of (4) is always negative, it is seen that viscosity tends to damp out any disturbance. On the other hand, a high rate of shear in the basic flow can cause the first term to be highly positive, thus fostering the growth of a disturbance. The relative importance of these two terms, then, determines the stability of the flow (*cf.* also the discussion in [*14*], § 4.5).

Another criterion of the same sort arises when the right-hand side of (4) is written in slightly different form. Indeed, since $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} = \operatorname{div}\left[(\boldsymbol{u} \cdot \boldsymbol{v})\, \boldsymbol{u}\right] - \boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{u} \cdot \boldsymbol{v},$$

---

[2] Several alternate and occasionally useful forms of (4) arise from the simple identity

$$\int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} = \int (\operatorname{curl} \boldsymbol{u})^2 = 2 \int \boldsymbol{D}' : \boldsymbol{D}',$$

where $\boldsymbol{D}'$ denotes the deformation matrix of the perturbation motion.

whence, by application of the divergence theorem, (4) can be written in the form

$$\boxed{\frac{d\mathcal{K}}{dt} = \int (\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{u} \cdot \boldsymbol{v} - \nu \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}).} \tag{7}$$

If the first term on the right of (7) is less than the second for all admissible vectors $\boldsymbol{u}$, then clearly the basic flow will be stable. But the size of the first term is governed by the magnitude of $\boldsymbol{v}$, from which it follows that high speeds in the basic flow, as well as high rates of shear, tend to cause instability. The qualitative nature of these effects will be investigated in the next several paragraphs.

It is important to note that the energy method cannot provide accurate knowledge of the limits of stability, such as can be gained from the linearized perturbation theory ([6], Part II; [14], Chap. 1). The reason is that in the energy method one establishes stability relative to arbitrary disturbances, while in reality only those satisfying the hydrodynamical equations need be considered. Nevertheless, because the energy method gives insight into the physical situation, and because the results have the merit of simplicity and complete mathematical rigor, the investigations based on it are both interesting and valuable.

With these preliminary remarks aside, we may now turn to the main result of the paper.

Theorem 1. *Let $\mathcal{V} = \mathcal{V}(t)$ be a bounded region of space, which can be included in some cube of edge length d. Let $\boldsymbol{v}$ be the velocity vector of any motion in $\mathcal{V}$ satisfying prescribed conditions at the boundary of $\mathcal{V}$. Then the kinetic energy of an arbitrary disturbance motion $\boldsymbol{u} = \boldsymbol{v}' - \boldsymbol{v}$ satisfies the inequalities*

$$\mathcal{K} \leqq \mathcal{K}_0\, e^{2(m - \alpha\nu/d^2)t}, \tag{8}$$

$$\mathcal{K} \leqq \mathcal{K}_0\, e^{(V^2 - \alpha\nu^2/d^2)t/\nu}. \tag{9}$$

Here $\mathcal{K}_0$ is the initial energy of the disturbance, $-m$ is a lower bound for the characteristic values of the deformation matrix of the basic flow over the time interval 0 to t, V is the maximum speed of the basic flow in the same time interval, and α is a pure number,

$$\alpha = \frac{3 + \sqrt{13}}{2}\,\pi^2 \cong 32.6. \tag{10}$$

This theorem is a generalization and improvement of certain results of T. Y. THOMAS and EBERHARD HOPF[3]. Before giving the proof of the theorem, we note two important corollaries, which in a certain sense constitute the heart of the paper. It is to be emphasized that these corollaries are absolutely rigorous, and apply to *all possible motions* in $\mathcal{V}$.

Universal stability criterion I. *If the dimensionless "Reynolds number" md^2/ν of a flow is less than 32.6, then $\mathcal{K} \to 0$ as $t \to \infty$, and the motion is stable.*

[3] THOMAS [8] proved that an arbitrary fluid motion is stable if its deformation matrix is sufficiently small, but did not compute numerical values; the present analysis is an extension of THOMAS' work. Similarly, HOPF has shown (essentially) that a fluid motion is stable if its maximum speed is small enough, again with no discussion of numerical values.

Universal stability criterion II. *If the Reynolds number Vd/ν of a flow is less than* 5.71, *then $\mathscr{K} \to 0$ as $t \to \infty$, and the motion is stable. (Here V is the maximum speed of the flow, and d is the maximum diameter of the flow region.)*

Proof of Theorem 1. We begin with an auxiliary computation whose goal is the inequality

$$\alpha d^{-2} \int u^2 \leqq \int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}, \tag{11}$$

where α is given by (10). Let $\boldsymbol{h}$ be an arbitrary continuously differentiable vector field in $\mathscr{V}$. Then for any value of the constant ε,

$$\begin{aligned} 0 &\leqq (u_i h_k + u_{i,k} + \varepsilon u_{k,i})(u_i h_k + u_{i,k} + \varepsilon u_{k,i}) \\ &= (1+\varepsilon^2) u_{i,k} u_{i,k} + h^2 u^2 + h_i (u^2)_{,i} + 2\varepsilon (u_i h_k + u_{i,k}) u_{k,i}. \end{aligned} \tag{12}$$

(For convenience we use tensor rather than vector notation in carrying out this and several subsequent calculations.) Some of the terms on the right-hand side of (12) can be transformed as follows,

$$h_i (u^2)_{,i} = (u^2 h_i)_{,i} - u^2 h_{i,i}$$

$$u_i h_k u_{k,i} = (u_i h_k u_k)_{,i} - u_i h_{k,i} u_k$$

$$u_{i,k} u_{k,i} = (u_i u_{k,i})_{,k}.$$

Here we have made use of the incompressibility condition $u_{i,i} = 0$. Making the above changes in (12), integrating the result over $\mathscr{V}$, and using the divergence theorem, we obtain the inequality

$$\int [(h_{i,i} - h^2) u^2 + 2\varepsilon u_i h_{i,k} u_k] \leqq (1+\varepsilon^2) \int u_{i,k} u_{i,k}. \tag{13}$$

Now the particular vector field $h_i = C \tan C x_i$ is differentiable in a cube of edge length π/C centered at the origin. If we set $C = \pi/d$ and suitably locate the origin in $\mathscr{V}$, then this vector can be substituted into (13). An easy computation then yields the inequality

$$(1+\varepsilon^2)^{-1} (3 + 2\varepsilon) C^2 \int u^2 \leqq \int u_{i,k} u_{i,k}.$$

The left-hand side is maximized by choosing $\varepsilon = \frac{1}{2}(\sqrt{13} - 3)$, and (11) is thereby proved.

The term $\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u}$ in (4) satisfies the inequality

$$\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} \geqq -m u^2 \tag{14}$$

during the entire time interval 0 to t. Therefore, from (4), (11) and (14)

$$\frac{d\mathscr{K}}{dt} \leqq (m - \alpha \nu d^{-2}) \int u^2 = 2(m - \alpha \nu d^{-2}) \mathscr{K}.$$

Writing this in the form

$$\frac{d}{dt} \{\mathscr{K} e^{-2(m - \alpha \nu d^{-2}) t}\} \leqq 0$$

and integrating from 0 to t leads at once to (8).

The proof of (9) is similar but requires in addition the inequality

$$\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{u} \cdot \boldsymbol{v} \leqq \frac{1}{2}\left(\nu \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} + \frac{u^2 v^2}{\nu}\right), \tag{15}$$

which follows at once from the identity

$$\boldsymbol{A}:\boldsymbol{A} - 2\boldsymbol{u}\cdot\boldsymbol{A}\cdot\boldsymbol{v} + u^2 v^2 \equiv (\boldsymbol{A} - \boldsymbol{u}\boldsymbol{v}) : (\boldsymbol{A} - \boldsymbol{u}\boldsymbol{v}) \geqq 0$$

with $\boldsymbol{A} = \nu \operatorname{grad} \boldsymbol{u}$. From (7), (15), and (11) we obtain

$$\frac{d\mathscr{K}}{dt} \leqq \frac{1}{\nu}\left(V^2 - \alpha \nu^2 d^{-2}\right)\mathscr{K},$$

and (9) then follows by integration as in the proof of (8).

3. Extensions and applications of Theorem 1

The preceding method can be applied equally well to flows which take place in channels and pipes, and to plane flow problems, so long as the fundamental formula (4) applies. To illustrate the method, consider for example a straight pipe whose cross section has a maximum diameter d. If the pipe is directed along the z-axis, then the vector

$$\boldsymbol{h} = C(\boldsymbol{i} \tan C x + \boldsymbol{j} \tan C y), \qquad (C = \pi/d),$$

can be used in (13). Choosing $\varepsilon = 0$, we get an inequality of the form (11) with $\alpha = 2\pi^2$. By using this result, the reader can easily obtain estimates for the rate of change of $\mathscr{K}$, and thus formulate analogues of Theorem 1 and the universal stability criteria for flow in a pipe.

Other cases can be treated similarly and the results conveniently grouped in the following table.

A. Straight channel, maximum width d:	$\alpha = \pi^2$.
B. Straight pipe, maximum diameter d:	$\alpha = 2\pi^2$.
C. Plane flow in a bounded region, maximum diameter d,	
i) Three dimensional disturbances:	$\alpha = 2\pi^2$,
ii) Plane disturbances only:	$\alpha = (1 + \sqrt{2})\,\pi^2$.
D. Bounded region, maximum diameter d:	$\alpha = \frac{3+\sqrt{13}}{2}\,\pi^2$.

(Note: It is only in cases C, ii) and D that we have been able to make use of the incompressibility condition $\operatorname{div} \boldsymbol{u} = 0$.)

We conclude the section with two simple applications of Theorem 1. First, suppose the boundaries of $\mathscr{V}$ consist of *rigid fixed* walls, so that any motion initially present will presumably die out due to lack of an energy source. By choosing $\boldsymbol{v} \equiv 0$ for the basic motion, we see from (8) that the energy $\mathscr{K}$ of an arbitrary motion $\boldsymbol{v}'$ in $\mathscr{V}$ must in fact tend to zero according to the law

$$\mathscr{K} \leqq \mathscr{K}_0\, e^{-2\alpha\nu t/d^2}, \qquad \alpha \cong 32.6.$$

Similar estimates, but with smaller values for the coefficient α, have been obtained by LERAY (for plane flow) and KAMPÉ DE FERIET[4] and BERKER (for spatial

[4] KAMPÉ DE FÉRIET obtained the value $\alpha = 3\pi^2$, using a method somewhat similar to the one presented here.

flows), and RAYLEIGH in a much earlier paper [4] proved that $\mathscr{K}\to 0$ as $t\to\infty$, though without estimating the rate of convergence. It remains an open question whether the velocity itself must tend to zero as $t\to\infty$; certainly one would expect this, but a strict proof seems to be a matter of more than ordinary difficulty.

As a second application, we have the following uniqueness theorem concerning *steady* motion in a fixed bounded region $\mathscr{V}$.

Theorem 2. *Let $\boldsymbol{v}$ and $\boldsymbol{v}'$ be two steady flows in $\mathscr{V}$, subject to a prescribed velocity distribution on the boundary of $\mathscr{V}$. Let $-m$ be a lower bound for the characteristic values of the deformation matrix of the motion $\boldsymbol{v}$, let $V=\max v$, and suppose that either*

$$m\, d^2/\nu \leqq \alpha \quad \text{or} \quad V\, d/\nu \leqq \sqrt{\alpha}. \tag{16}$$

Then the two flows are identical.

Proof. The kinetic energy of the difference motion $\boldsymbol{v}'-\boldsymbol{v}$ must be constant, since the flows are steady. On the other hand, it must satisfy both (8) and (9). In view of (16) this can happen only if $\mathscr{K}=0$, which in turn implies $\boldsymbol{v}=\boldsymbol{v}'$.

This theorem depends strongly on the condition (16), but without some such assumption it is extremely unlikely that the conclusion is true.

4. Variational techniques

The key step in the proof of Theorem 1 lay in establishing inequality (11). We are concerned, however, not only with the validity of (11), but also with the determination of the largest possible coefficient α, for the size of α evidently determines the numerical values in the various stability criteria. Now the method of proof in Theorem 1 clearly gives no guarantee of providing the "best possible" value for α; moreover, it is quite crude in its estimate of the term $\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u}$. For these reasons, it is of great interest to consider an alternate approach which not only supplies a theoretical procedure for determining the best possible α, but in addition gives a way to avoid the estimate (14).

Stated in precise terms, our problem is twofold. First, we must determine the greatest coefficient α such that the inequality

$$\alpha\, d^{-2} \int u^2 \leqq \int \operatorname{grad}\boldsymbol{u} : \operatorname{grad}\boldsymbol{u}$$

holds for arbitrary vector fields $\boldsymbol{u}$ satisfying

$$\operatorname{div}\boldsymbol{u}=0, \qquad \boldsymbol{u}=0 \text{ on } \mathscr{S}. \tag{17}$$

Second, we must determine the least coefficient $\tilde{\nu}$ such that the inequality

$$\int (\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u} + \tilde{\nu}\operatorname{grad}\boldsymbol{u} : \operatorname{grad}\boldsymbol{u}) \geqq 0$$

holds, again for arbitrary $\boldsymbol{u}$ satisfying (17). In this case it is clear that the basic flow will be stable provided simply that $\nu>\tilde{\nu}$. The two problems above can be consolidated into the single variational problem:

$$-\int \boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u} = \text{Maximum}, \tag{18}$$

where the vector field $\boldsymbol{u}$ must satisfy the side conditions

$$\int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} = 1, \qquad \operatorname{div} \boldsymbol{u} = 0, \qquad \boldsymbol{u} = 0 \text{ on } \mathscr{S}. \tag{19}$$

The first problem occurs when $\boldsymbol{D}$ is the negative of the identity matrix.

The variational problem (18)—(19) can be reformulated as a partial differential equation for the extremal function $\boldsymbol{u}$, according to well known procedures of the calculus of variations. Thus, through introduction of Lagrange multipliers ν^* and $\lambda = \lambda(x, y, z, t)$, the problem becomes

$$\delta \int (\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} + \nu^* \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} - 2\lambda \operatorname{div} \boldsymbol{u}) = 0. \tag{20}$$

The Euler-Lagrange equation corresponding to (20) is easily found to be

$$\boldsymbol{u} \cdot \boldsymbol{D} = -\operatorname{grad} \lambda + \nu^* \nabla^2 \boldsymbol{u}, \tag{21}$$

and this is to be solved subject to the side conditions (19). The reader may observe the remarkable similarity between the eigenvalue equation (21) and the equations of hydrodynamics. The equations corresponding to the case $\boldsymbol{D} = -\boldsymbol{I}$ are of sufficient importance to be noted separately, namely

$$\nu^* \nabla^2 \boldsymbol{u} + \boldsymbol{u} = \operatorname{grad} \lambda, \qquad \operatorname{div} \boldsymbol{u} = 0. \tag{22}$$

In spite of the relative simplicity of (22), we have been unable to determine its solution, and in what follows can only offer some general remarks concerning the system (19), (21).

Now for any solution of equation (21) we have

$$-\int \boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} = \int (\boldsymbol{u} \cdot \operatorname{grad} \lambda - \nu^* \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u}) = \nu^* \int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} = \nu^*,$$

where use has been made of the divergence theorem and the conditions (19). On the other hand, any vector $\boldsymbol{u}$ which provides the integral (18) with its maximum value must be a solution of (21). It follows that the eigenvalue $\tilde{\nu}$ associated with a maximizing vector is precisely the maximum value of the integral (18). Moreover, no eigenvalue ν^* can be larger than $\tilde{\nu}$, for if this were the case then the corresponding eigenvector $\boldsymbol{u}^*$ would give to the integral (18) a value larger than $\tilde{\nu}$. This proves the following theorem.

Theorem 3. *Suppose there exists a vector* $\boldsymbol{u}$ *which solves the variational problem* (18)—(19). *Then the eigenvalue problem* (19), (21) *has a greatest eigenvalue* $\tilde{\nu}$, *and the basic motion will be stable provided that* $\nu > \tilde{\nu}$.

The difficulty with Theorem 3 is, of course, that there is no direct way of verifying its hypothesis. In circumstances such as these the best that can be hoped for is to determine all the eigenvalues (by Theorem 1 they constitute a bounded set), and to assume that the least upper bound of these eigenvalues is the required maximum of the integral (18). If the eigenvectors are complete in the set of admissible vectors, then this gives another method for proving that the least upper bound of the eigenvalues is the maximum of (18); unfortunately, this appears at least as difficult to verify as the hypothesis of Theorem 3.

5. Example. Couette flow

We consider the well known circulatory flow between rotating concentric cylinders. If the inner and outer cylinders, respectively, have radii R_1 and R_2, and rotate with angular speeds Ω_1 and Ω_2 ($\Omega_1 > 0$), then the velocity field is given by

$$v_\vartheta = A r + B r^{-1}, \qquad v_r = v_z = 0,$$

where

$$A = \frac{R_2^2 \Omega_2 - R_1^2 \Omega_1}{R_2^2 - R_1^2}, \qquad B = -\frac{(R_1 R_2)^2 (\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}.$$

One finds easily that the scalar vorticity ω has the constant value $2A$, while in polar coordinates

$$\boldsymbol{D} = -\frac{B}{r^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic values of $\boldsymbol{D}$ are $\pm B/r^2$, whence

$$\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} \geqq -\frac{|B|}{r^2} u^2. \tag{23}$$

The special form of inequality (23) suggests that the stability estimates of § 2 can be considerably improved. In particular, in place of the inequality (11) we shall look for an estimate of the form

$$\beta \int \frac{u^2}{r^2} \leqq \int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}. \tag{24}$$

To this end, let us seek a vector field $\boldsymbol{h}$ such that

$$\operatorname{div} \boldsymbol{h} - h^2 = C^2/r^2, \qquad R_1 < r < R_2. \tag{25}$$

A radially symmetric solution of (25) is readily found, namely

$$\boldsymbol{h} = \frac{C}{r} \tan(C \log r + D)\, \boldsymbol{i}_r, \tag{26}$$

where the constants C and D are given by

$$C = \frac{\pi}{\log(R_2/R_1)}, \qquad D = \frac{\pi}{2}\, \frac{\log(R_1 R_2)}{\log(R_2/R_1)}.$$

Substituting (26) into (13) and setting $\varepsilon = 0$ then yields an inequality of the form (24), with

$$\beta = C^2 = \left\{\frac{\pi}{\log(R_2/R_1)}\right\}^2.$$

Combining (23) and (24) with the fundamental energy equation (4) yields

$$\frac{d\mathscr{K}}{dt} \leqq (|B| - \beta \nu) \int \frac{u^2}{r^2},$$

and from this it follows that *Couette flow is stable relative to arbitrary disturbances whenever*

$$\left|\frac{\Omega_2 - \Omega_1}{\nu}\right| < (R_2^2 - R_1^2) \left\{\frac{\pi}{R_1 R_2 \log(R_2/R_1)}\right\}^2. \tag{27}$$

The reader should notice that this stability is in the "strong" sense, $\mathscr{K} \to 0$ as $t \to \infty$.

Previously, SYNGE [*13*] has proved stability relative to infinitesimal disturbances whenever A has the same sign as Ω_1, which in the present case ($\Omega_1 > 0$) gives the stability conditions $\omega = 2A \geqq 0$, or equivalently, $\Omega_2 \geqq (R_2/R_1)^2 \Omega_1$ [5]. The relative zones of stability are shown in Fig. 1. For the celebrated pair of radii $R_1 = 3.55$, $R_2 = 4.03$, inequality (27) reduces to

$$\left| \frac{\Omega_2 - \Omega_1}{\nu} \right| < 10.92. \tag{28}$$

In Fig. 2 we have indicated the stability zones based on (28), on SYNGE's criterion, and on the calculations and experiments of G. I. TAYLOR. If we suppose TAYLOR's

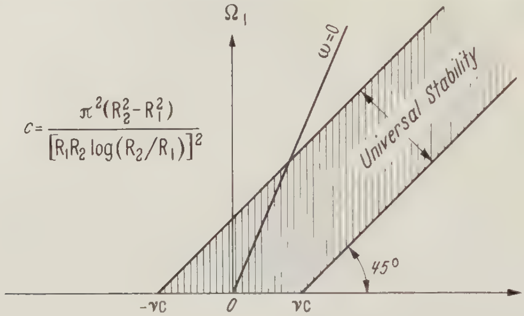

Fig. 1. Stability zones for Couette flow. The zone $\omega \geqq 0$ lies *below* the line $\omega = 0$

experimental data to be an accurate representation of the mathematical situation, it is seen that the right-hand side of (28) cannot possibly be greater than about 50, for otherwise there is an observed secondary motion. Our value is certainly not

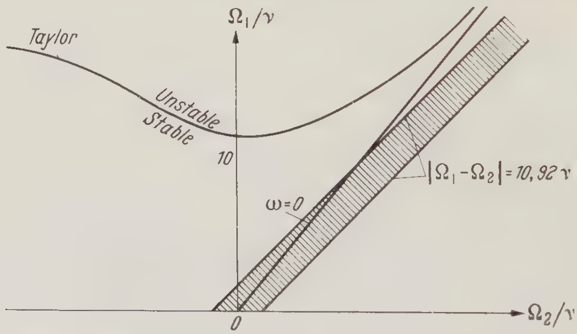

Fig. 2. Stability zones for Couette flow, $R_1 = 3.55$ and $R_2 = 4.03$

very near 50, but considering the difficulty of the problem and the fact that (28) applies to arbitrary disturbances, one does feel that 10.92 is a quite respectable value.

6. Couette flow (continued)

In the previous section we studied the stability of Couette flow using the methods of § 2. In this section the same problem will be treated by the variational techniques of § 4. The end result will be seen to have a somewhat tentative

[5] In this connection we may recall some earlier work of RAYLEIGH and SYNGE, in which the same criterion is established for inviscid fluids; *cf.* reference [*14*], § 4.2.

character, in line with the remarks at the close of § 4, though it still has considerable interest. For Couette flow the differential equation (21) and the side condition div $\boldsymbol{u}=0$ take the following form (in polar coordinates)

$$\begin{aligned}
-\frac{Bv}{r^2} &= -\frac{\partial\lambda}{\partial r}+\nu^*\left\{\Delta u-\frac{2}{r^2}\frac{\partial v}{\partial\vartheta}-\frac{u}{r^2}\right\},\\
-\frac{Bu}{r^2} &= -\frac{1}{r}\frac{\partial\lambda}{\partial\vartheta}+\nu^*\left\{\Delta v+\frac{2}{r^2}\frac{\partial u}{\partial\vartheta}-\frac{v}{r^2}\right\},\\
0 &= -\frac{\partial\lambda}{\partial z}+\nu^*\Delta w,\\
&\frac{1}{r}\frac{\partial}{\partial r}(r u)+\frac{1}{r}\frac{\partial v}{\partial\vartheta}+\frac{\partial w}{\partial z}=0,
\end{aligned} \tag{29}$$

where

$$\Delta\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}+\frac{\partial^2}{\partial z^2},$$

and u, v, w denote, respectively, the r, ϑ and z components of the perturbation velocity $\boldsymbol{u}$. Equations (29), as they stand, are too difficult to solve in full generality. We shall therefore seek a particular solution of the form

$$u=\hat{u}(r)\cos kz,\qquad v=\hat{v}(r)\cos kz,\qquad w=\hat{w}(r)\sin kz. \tag{30}$$

Eliminating λ and $\hat{w}$ from the system (29), we get

$$\begin{aligned}
(L-k^2)^2\hat{u} &= \frac{Bk^2}{\nu^*}\frac{\hat{v}}{r^2},\\
(L-k^2)\,\hat{v} &= -\frac{B}{\nu^*}\frac{\hat{u}}{r^2},
\end{aligned} \tag{31}$$

where L denotes the differential operator

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)-\frac{1}{r^2}.$$

(The derivation of (31) is most simply carried out as follows. From the last equation of (29) we have

$$\frac{1}{r}\frac{d}{dr}(r\hat{u})+k\hat{w}=0,$$

whence $\boldsymbol{u}$ can be written in the form $\boldsymbol{u}=\operatorname{curl}\boldsymbol{\Psi}$, where

$$\boldsymbol{\Psi}=\frac{1}{k}(\hat{v}\,\boldsymbol{i}_r-\hat{u}\,\boldsymbol{i}_\vartheta)\sin kz.$$

Substituting $\boldsymbol{u}=\operatorname{curl}\boldsymbol{\Psi}$ directly into (21) and taking the curl of that equation yields

$$B\operatorname{curl}(v/r^2,\,u/r^2,\,0)=\nu^*\operatorname{curl}^4\boldsymbol{\Psi}, \tag{32}$$

where the identity curl curl $=$ grad div $-\nabla^2$ has been used. The first equation of (31) is now obtained as the ϑ-component of equation (32). Finally, the unkown λ must necessarily be independent of ϑ, so that the second equation of (29) reduces immediately to the second equation of (31).)

Equations (31) are formally similar to the classical first order equations for small disturbance in Couette flow. The solution of (31) is a difficult task unless the cylinders are close together, in which case we can suppose (31) to be sufficiently well approximated by the simpler equations

$$\begin{aligned}(D^2-k^2)^2\hat{u} &= \Lambda k^2 \hat{v}\\ (D^2-k^2)\,\hat{v} &= -\Lambda \hat{u},\end{aligned} \tag{33}$$

where $D=d/dr$, $\Lambda=B/\nu^*\bar{R}^2$, and $\bar{R}$ is some appropriate mean radius, say $\bar{R}=\sqrt{R_1 R_2}$. The boundary conditions associated with (33) are

$$\hat{u}=\hat{v}=\frac{d\hat{u}}{dr}=0 \quad \text{at} \quad r=R_1, R_2. \tag{34}$$

According to the calculations of JEFFREYS and others (*cf.* [*14*], § 2.3) the critical value of Λ is $41.2\,(R_2-R_1)^{-2}$ (that is, this is the smallest value of Λ for which a non-trivial solution of (33), (34) exists). We are thus led to the following stability criterion,

$$\left|\frac{\Omega_2-\Omega_1}{\nu}\right| \leqq 41.2\,\frac{R_1+R_2}{R_1 R_2 (R_2-R_1)}. \tag{35}$$

For the pair of radii 3.55, 4.03 this gives, in particular,

$$\left|\frac{\Omega_2-\Omega_1}{\nu}\right| \leqq 45.5. \tag{36}$$

(36) is certainly a real improvement over the earlier estimate (28), and, moreover, the right-hand side comes very close to the "best value" 50. It must be borne in mind, however, that (35) and (36) are not yet rigorously established, since it remains to be shown that we have really found the greatest eigenvalue of the system (29). There is some reason to believe that this is so, since hydrodynamical experiments have shown indisputably that the secondary motion occurring in Couette flow is approximately of the form (30), but barring a proof of this fact, we must accept (35)—(36) as only tentatively established, in contrast with the absolute certainty of (27)—(28).

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References

[*1*] REYNOLDS, O.: On the dynamical theory of incompressible viscous fluids and the determination of the criterion. Phil. Trans. Roy. Soc. London (A) **186**, 123—164 (1895); Scientific Papers II, pp. 535—577.

[*2*] ORR, W. McF.: The stability or instability of the steady motions of a liquid. Part II: A viscous liquid. Proc. Roy. Irish Acad. (A) **27**, 69—138 (1907).

[*3*] HAMEL, G.: Zum Turbulenzproblem. Nachr. Ges. Wiss. Göttingen. **1911**, 261—270.

[*4*] LORD RAYLEIGH: On the motion of a viscous fluid. Phil. Mag. (6) **26**, 776—786 (1913); Papers VI, pp. 187—197.

[*5*] KÁRMÁN, TH. VON: Über die Stabilität der Laminarströmung und die Theorie der Turbulenz. Proc. 1st. Int. Congress Appl. Mech. Delft, 1924, pp. 97—112.

[*6*] SYNGE, J. L.: Hydrodynamical stability. Semi-centennial publications of the Amer. Math. Soc. **2**, 227—269 (1938). Especially Part III.

[7] Thomas, T. Y.: On the stability of viscous fluids. (1941). Univ. Calif. Publ. Math. New Series **2**, 13—43 (1944).
[8] Thomas, T. Y.: On the uniform convergence of the solutions of the Navier-Stokes equations. Proc. Nat. Acad. Sci. U.S.A. **29**, 243—246 (1943).
[9] Hopf, E.: On non-linear partial differential equations. Lecture series of the Symposium on partial differential equations. Univ. of Calif. 1955, pp. 7—11.
[10] Leray, J.: Essai sur les mouvements plan d'un liquide visqueux que limitent des parois. J. Math. Pures. Appl. (9) **13**, 331—418 (1934).
[11] Kampé de Fériet, J.: Sur la decroissance de l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine borné ayant pour frontière des parois solides fixes. Ann. Soc. Sci. Bruxelles **63**, 35—46 (1949).
[12] Berker, R.: Inégalité vérifiée par l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine spatial borné. Bull. Tech. Univ. Istanbul **2**, 41—50 (1949).
[13] Synge, J. L.: On the stability of a viscous liquid between two rotating coaxial cylinders. Proc. Roy. Soc. London (A) **167**, 250—256 (1938).
[14] Lin, C. C.: The theory of hydrodynamic stability. Cambridge 1955. Especially Chaps. 1, 2, 4.

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Rayleigh's Problem in Hydromagnetics: The Impulsive Motion of a Pole-piece

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Introduction

Consider the flow of an inviscid fluid past the pole-piece of a permanent magnet, *i.e.*, a perfectly conducting rigid body containing a magnetic field. If the fluid is non-conducting, the field of the magnet exerts no force on it, and, as is usual in the ordinary theory of inviscid fluids, no condition on the tangential component of velocity at the pole-piece need be satisfied. At the other extreme, when the fluid is a perfect conductor also, its velocity, $\boldsymbol{v}$, must vanish at any point of the pole-piece at which the normal component of the magnetic induction $\boldsymbol{B}$ is non-zero*. It follows that any initial discontinuity in velocity at the boundary must immediately move away**, in contrast to the case of a non-conducting fluid, where the discontinuity can cling to the pole-piece as a vortex sheet.

The object of this paper is to give a simple exact solution for which the transition from zero to infinite conductivity can be traced and the modifying effects of viscosity determined. Thus we consider the motion of an incompressible, viscous, electrically conducting fluid contained between the parallel, plane pole-pieces, $y = 0, h$, of a permanent magnet which provides a uniform external field in the y-direction. Starting at time $t = 0$, with the fluid at rest, the magnet is made to move uniformly with velocity u_0 in the negative x-direction.

The limiting case of zero conductivity and infinite separation ($h = \infty$) was considered by Rayleigh [*1*]; the fluid velocity distribution is then given by

$$u = u_0 \operatorname{erf}\left(y/2\sqrt{\nu t}\right), \tag{1}$$

where ν is the kinematic viscosity and the axes move with the magnet. Recently, Rossow [*3*] has extended the discussion to a conducting fluid, but his equations are inexact (see Section 5). In the present paper the solution of the general problem is obtained by means of the Laplace transform; in certain cases it can be expressed by known functions.

* The tangential component of the electric field $\boldsymbol{E}$ is continuous, and hence zero, at the interface, while $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$ is zero everywhere in a perfectly conducting fluid.

** For the resolution of such shear flow discontinuities in a compressible perfectly conducting fluid see [*6*].

Infinite separation of the pole-pieces is considered first. The boundary-layer solution for vanishingly small viscosity gives, at any time t, a velocity profile which is similar to (1), the similarity factor being an exponentially decreasing function of t. This transport of vorticity out of the boundary layer becomes more rapid as the conductivity increases, the vorticity tending to become concentrated in a diffused Alfvén wave. This is borne out by the special case of equal, but not necessarily small, diffusion coefficients, for which the solution is given by known functions, and the case of small and comparable coefficients, when the principal effects can be determined by the method of steepest descent. Finally, we give general series expansions which are particularly suited to the initial stages of the motion, and also discuss the character of the flow for large values of t.

The flow with finite separation of the pole-pieces is the sum of a series of flows, each of which is of the kind described above; the same result is obtained by the method of images. It immediately follows, in particular, that each of the two Alfvén waves which leave the pole-pieces when the viscosity and resistivity are small, is repeatedly reflected, first at one and then at the other of the two pole-pieces.

Whenever the diffusion coefficients are not both zero, the magnetic disturbance dies out as $t \to \infty$. For infinite separation, however, there is a residual magnetic field, its value being independent of these coefficients. This is of importance in calculating the shearing stress on a pole-piece, since the Maxwell stress contributes. Thus, the assumption that h may be replaced by ∞ for widely spaced pole-pieces leads to completely wrong estimates when t is sufficiently large.

At all stages of the motion the shearing stress is larger than

(2) $$\tau = \varrho_0 u_0 \sqrt{\nu/\pi t},$$

the value obtained in Rayleigh's case, although the viscous part is smaller.

1. Equations of Motion

The fluid is assumed to be incompressible, viscous, and electrically conducting, and to fill the space between the pole-pieces $y = 0$, h of a magnet which is supposed to be a perfect conductor. All quantities are functions of y and t only, so that if the velocity $\boldsymbol{v}$ and magnetic field $\boldsymbol{H}$ are given by $(u, v, 0)$ and $(H, H_0, 0)$, respectively, we find*:

(i) from the equation of continuity: $\operatorname{div} \boldsymbol{v} = \partial v/\partial y = 0$, so that v must be zero since this is its value at either pole-piece;

(ii) from $\operatorname{div} \boldsymbol{H} = \partial H_0/\partial y = 0$, it follows that H_0 has at each instant its value at either pole-piece, which, by continuity of normal induction, is constant;

(iii) since the current $\boldsymbol{J}$ is given by curl $\boldsymbol{H}$ it follows from the conduction equation:

(3) $$\boldsymbol{J} = \sigma(\boldsymbol{E} + \mu\, \boldsymbol{v} \times \boldsymbol{H}),$$

that the electric field $\boldsymbol{E}$ acts in the z-direction: $\boldsymbol{E} = (0, 0, E)$.

* Displacement currents will be neglected and the material coefficients ν (viscosity), μ (permeability), and σ (conductivity) assumed constant.

The equations governing the motion are therefore the x-component of Maxwell's equation $\operatorname{curl}\boldsymbol{E} = -\mu\,\partial\boldsymbol{H}/\partial t$, *viz.*

$$\frac{\partial E}{\partial y} = -\mu\frac{\partial H}{\partial t}, \tag{4}$$

the z-component of (3),

$$-\frac{\partial H}{\partial y} = \sigma(E + \mu u H_0), \tag{5}$$

and the x- and y-components of the momentum equation,

$$\varrho_0\frac{\partial u}{\partial t} = \mu H_0\frac{\partial H}{\partial y} + \varrho_0\nu\frac{\partial^2 u}{\partial y^2}, \tag{6}$$

$$0 = -\frac{\partial \varpi}{\partial y} - \mu H\frac{\partial H}{\partial y}, \tag{7}$$

where ϖ is the pressure and ϱ_0 the density. The remaining components of these equations are satisfied identically.

Equation (7) integrates immediately to show that $\varpi - \mu H^2/2$ is constant at each instant, while on eliminating E from (4), (5), and (6) we find that H and u satisfy

$$\begin{aligned}\left(\eta\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right)H + H_0\frac{\partial u}{\partial y} &= 0,\\ A_0^2\frac{\partial H}{\partial y} + H_0\left(\nu\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right)u &= 0,\end{aligned} \tag{8}$$

where $A_0 = \sqrt{\mu H_0^2/\varrho_0}$ is the Alfvén velocity and $\eta = 1/\mu\sigma$ is the magnetic diffusivity. The initial and boundary conditions on the solution of this system are*

$$t = 0: \qquad H = 0, \quad u = u_0 \quad \text{for all } y, \tag{9}$$

$$y = 0, h: \qquad \frac{\partial H}{\partial y} = u = 0 \quad \text{for all } t, \tag{10}$$

the latter being equivalent to the vanishing of E and u, see (5). Since the magnet is a perfect conductor there is no restriction on H itself at a pole-piece; any discontinuity in the tangential field corresponds to surface currents (in the z-direction).

2. Laplace Transform of Solution

From (8) and (9) the Laplace transforms of H and u satisfy the equations

$$\begin{aligned}\left(\eta\frac{d^2}{dy^2} - p\right)H + H_0\frac{du}{dy} &= 0,\\ A_0^2\frac{dH}{dy} + H_0\left(\nu\frac{d^2}{dy^2} - p\right)u &= -H_0 u_0,\end{aligned} \tag{11}$$

where p is the transform parameter and the same symbols are used for the transforms. The general solution of this system is

$$H = A\,e^{my} + B\,e^{-my} + C\,e^{ny} + D\,e^{-ny},$$

$$u = \frac{u_0}{p} + \frac{p - \eta m^2}{H_0 m}\left[A\,e^{my} - B\,e^{-my}\right] + \frac{p - \eta n^2}{H_0 n}\left[C\,e^{ny} - D\,e^{-ny}\right],$$

* As in the Introduction, the axes move with the magnet.

where A, B, C, and D are arbitrary constants, and $r = \pm m$, $\pm n$ are the roots of

$$(\eta r^2 - p)(\nu r^2 - p) - A_0^2 r^2 = 0. \tag{12}$$

We take

$$m = \sqrt{a + b p} + \sqrt{a + c p}, \qquad n = \sqrt{a + b p} - \sqrt{a + c p},$$

where

$$a = \frac{A_0^2}{4 \eta \nu}, \qquad b = \frac{(\sqrt{\eta} + \sqrt{\nu})^2}{4 \eta \nu}, \qquad c = \frac{(\sqrt{\eta} - \sqrt{\nu})^2}{4 \eta \nu}.$$

The constants A, B, C, D must now be chosen so that H and u satisfy the boundary conditions (10). Thus we find

$$H = \frac{H_0 u_0}{p^2} \cdot \frac{m n}{m^2 - n^2} \left[n \frac{\sinh m (y - \frac{1}{2} h)}{\cosh \frac{1}{2} m h} - m \frac{\sinh n (y - \frac{1}{2} h)}{\cosh \frac{1}{2} n h} \right], \tag{13a}$$

$$\begin{aligned} u = {} & \frac{u_0}{p} + \frac{u_0}{p^2 (m^2 - n^2)} \times \\ & \times \left[n^2 (p - \eta m^2) \frac{\cosh m (y - \frac{1}{2} h)}{\cosh \frac{1}{2} m h} - m^2 (p - \eta n^2) \frac{\cosh n (y - \frac{1}{2} h)}{\cosh \frac{1}{2} n h} \right], \end{aligned} \tag{13b}$$

or, for the case $h = \infty$,

$$H = \frac{H_0 u_0}{p^2} \cdot \frac{m n}{m^2 - n^2} \left[- n e^{-m y} + m e^{-n y} \right], \tag{14a}$$

$$u = \frac{u_0}{p} + \frac{u_0}{p^2 (m^2 - n^2)} \left[n^2 (p - \eta m^2) e^{-m y} - m^2 (p - \eta n^2) e^{-n y} \right]. \tag{14b}$$

A somewhat simpler function is the current $J = - d H / d y$:

$$J = - \frac{H_0 u_0}{4 \eta \nu} \cdot \frac{1}{\sqrt{(a + b p)(a + c p)}} \left[\frac{\cosh m (y - \frac{1}{2} h)}{\cosh \frac{1}{2} m h} - \frac{\cosh n (y - \frac{1}{2} h)}{\cosh \frac{1}{2} n h} \right], \tag{15}$$

or, for h infinite,

$$J = \frac{H_0 u_0}{2 \eta \nu} \cdot \frac{1}{\sqrt{(a + b p)(a + c p)}} \, e^{-\sqrt{a + b p}\, y} \sinh \sqrt{a + c p}\, y. \tag{16}$$

3. Infinite Separation and the Special Case $\eta = \nu$

In the next four sections we shall be concerned with infinite separation of the pole-pieces: $h = \infty$. Even then none of the transforms H, u, or J can, in general, be inverted in terms of known functions. However, when $\eta = \nu = k$ (say), *i.e.* $c = 0$, this can be done; we find

$$J = \frac{H_0 u_0}{A_0} \cdot \frac{1}{2 \sqrt{\pi k t}} \left[\exp\left\{ - \frac{(y - A_0 t)^2}{4 k t} \right\} - \exp\left\{ - \frac{(y + A_0 t)^2}{4 k t} \right\} \right], \tag{17}$$

while

$$H = \frac{1}{2} \frac{H_0 u_0}{A_0} \left[- \operatorname{erf}\left(\frac{y - A_0 t}{2 \sqrt{k t}} \right) + \operatorname{erf}\left(\frac{y + A_0 t}{2 \sqrt{k t}} \right) \right],$$

$$u = \frac{1}{2} u_0 \left[\operatorname{erf}\left(\frac{y - A_0 t}{2 \sqrt{k t}} \right) + \operatorname{erf}\left(\frac{y + A_0 t}{2 \sqrt{k t}} \right) \right].$$

These functions are graphed in Figs. 1—3.

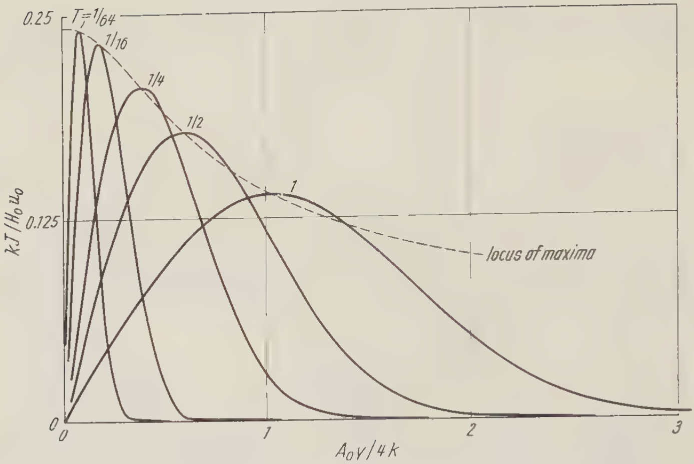

Fig. 1. Variation of current J with distance y for various values of $T=A_0^2\,t/4k$, when $\eta=\nu=k$

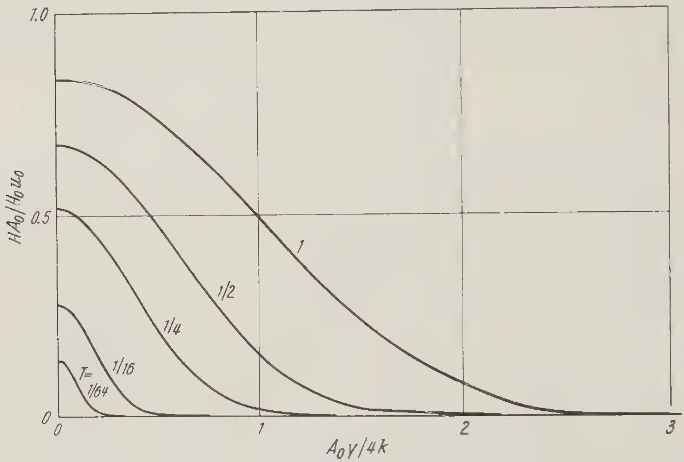

Fig. 2. Variation of the disturbance field H with distance y for various values of $T=A_0^2\,t/4k$, when $\nu=\eta=k$

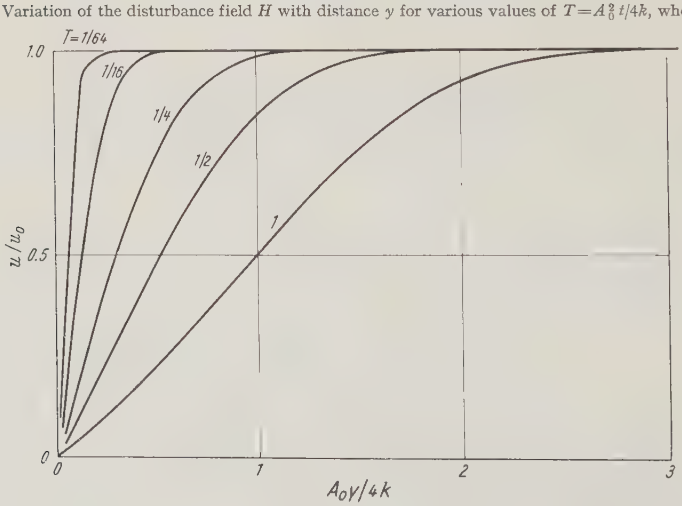

Fig. 3. Variation of the velocity u with distance y for various values of $T=A_0^2\,t/4k$, when $\nu=\eta=k$

Clearly J is the total effect of two diffusion sources of strength $\pm H_0 u_0/A_0$ moving with velocities $\pm A_0$, the diffusion coefficient being k. In the limit $k \to 0$ it becomes (for $y > 0$) $H_0 u_0/A_0$ times the delta function $\delta(y - A_0 t)$, and then H and u are $H_0 u_0/A_0$ and u_0 times the Heaviside unit functions $\mathscr{H}(A_0 t - y)$ and $\mathscr{H}(y - A_0 t)$ respectively. The initial discontinuity moves away from the pole-pieces as an Alfvén wave which is both a current sheet and vortex sheet: H jumps from zero to $H_0 u_0/A_0$ while u jumps from u_0 to zero. It is a mistake to speak of this wave as a shock, however (*cf.* STEWARTSON [*4*] who obtains a similar limit in another problem), since it is a valid solution of (8), for $\eta = \nu = 0$, in which the variables have finite jumps across the characteristic $y = A_0 t$*.

It is easy to see why the equality of η and ν leads to simple results. The operator of the system (8) is then the product of the two operators

$$k \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t} \mp A_0 \frac{\partial}{\partial y},$$

and these are diffusion operators relative to axes moving with velocity $\pm A_0$.

When $\sqrt{\eta} - \sqrt{\nu}$ is small, but $\sqrt{\eta\nu}$ is not, successive approximations to the solution can be found by regular perturbation. Thus in (16), for example, we expand in c:

$$\frac{\exp\{[\sqrt{a+bp} \pm \sqrt{a+cp}]\, y\}}{\sqrt{(a+bp)(a+cp)}} = \frac{\exp\{[\sqrt{a+bp} \pm \sqrt{a}]\, y\}}{\sqrt{a(a+bp)}} [1 + c\bar{K} p + \cdots],$$

where $\bar{K} = -(1 \mp \sqrt{a}\, y)/2a$, and hence find (to the first order)

$$J = \frac{H_0 u_0}{A_0'} \left\{ \left(1 + K^+ \frac{\partial}{\partial t}\right) \left[\frac{1}{2\sqrt{\pi k t}} \exp\left\{ - \frac{(y - A_0' t)^2}{4kt} \right\} \right] - \right.$$
$$\left. - \left(1 + K^- \frac{\partial}{\partial t}\right) \left[\frac{1}{2\sqrt{\pi k t}} \exp\left\{ - \frac{(y + A_0' t)^2}{4kt} \right\} \right] \right\},$$

where

$$A_0' = \frac{4\sqrt{\eta\nu}}{(\sqrt{\eta} + \sqrt{\nu})^2} A_0, \qquad k = \frac{4\eta\nu}{(\sqrt{\eta} + \sqrt{\nu})^2}, \qquad K^\pm = -\frac{(\sqrt{\eta} - \sqrt{\nu})^2}{2A_0^2} \left[1 \mp \frac{A_0 y}{2\sqrt{\eta\nu}} \right].$$

The motion is therefore effectively the sum of a pair of diffused waves with reduced velocities $\pm A_0'$ and diffusion coefficient k, whose square root is the harmonic mean of the square roots of η and ν.

4. General Case with η and ν Small but Comparable

When η and ν are small, but otherwise quite general, an indication of the character of the motion can be obtained by the method of steepest decent. Setting $p = A_0^2 s/\sqrt{\eta\nu}$ in the Bromwich contour integral defining the inverse of (16), we have to evaluate

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \exp \vartheta \left[st - y\{\sqrt{1 + \alpha s} \pm \sqrt{1 + \beta s}\}/2A_0 \right] \frac{ds}{\sqrt{(1 + \alpha s)(1 + \beta s)}} \tag{18}$$

* See the remarks of VON MISES [*5*] on the similar question of contact discontinuities in ordinary hydrodynamics.

for large values of $\vartheta = A_0^2/\sqrt{\eta\nu}$. Here

$$\alpha = \sqrt{\frac{\eta}{\nu}} + \sqrt{\frac{\nu}{\eta}} + 2, \quad \beta = \sqrt{\frac{\eta}{\nu}} + \sqrt{\frac{\nu}{\eta}} - 2, \quad \alpha - \beta = 4,$$

and γ is any real constant $> -1/\alpha$.

Now the stationary points of the arguments of the exponential are determined by

$$\frac{\alpha}{\sqrt{1+\alpha s}} \pm \frac{\beta}{\sqrt{1+\beta s}} = 4\varkappa, \quad \varkappa = \frac{A_0 t}{y},$$

and with s the corresponding functions of $\varkappa$ the largest argument is obtained for $\varkappa = 1$ with the lower sign. Hence, for $\varkappa$ sufficiently close to unity, the difference of the integrals (18) has the asymptotic form

$$4i\sqrt{\frac{\pi}{\vartheta t}} \cdot \frac{\varkappa^{\frac{1}{2}}(1+\alpha s)^{\frac{1}{4}}(1+\beta s)^{\frac{1}{4}}}{[\alpha^2(1+\beta s)^{\frac{3}{2}} - \beta^2(1+\alpha s)^{\frac{3}{2}}]^{\frac{1}{2}}} \cdot \exp\vartheta t\left[s - \{\sqrt{1+\alpha s} - \sqrt{1+\beta s}\}/2\varkappa\right]$$

where

$$s = 2(1-\varkappa)/(\alpha+\beta) + O(1-\varkappa)^2,$$

and elsewhere is negligible.

It follows that J has the approximate asymptotic form

(19) $$J \sim \frac{H_0 u_0}{A_0} \cdot \frac{1}{\sqrt{2\pi(\eta+\nu)t}} \cdot \exp\left\{-\frac{(y-A_0 t)^2}{2(\eta+\nu)t}\right\},$$

which should be compared with the exact result (17) for $\eta = \nu$, and interpreted in a similar fashion; the effective diffusion coefficient is the arithmetic mean of η and ν.

5. The (Viscous) Boundary Layer and the Flow Outside

Clearly the above arguments do not apply when η and ν are not comparable, since then α and β are also large. Of particular interest in this respect is the case where ν is small compared to η.

For any fixed $\eta \neq 0$, the boundary-layer solution at the pole-piece is obtained by letting $\nu \to 0$ with $y/\sqrt{\nu}$ fixed. In the limit we find

(20) $$\sqrt{\nu}\, m = \sqrt{p + A_0^2/\eta}, \quad n = \frac{p}{\sqrt{\eta p + A_0^2}},$$

so that

(21) $$J = \frac{H_0 u_0}{\eta} e^{-A_0^2 t/\eta} \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

while from (14)

(22) $$H = \frac{H_0 u_0}{A_0} \operatorname{erf}\left(A_0\sqrt{\frac{t}{\eta}}\right), \quad u = u_0 e^{-A_0^2 t/\eta} \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)^{\star}.$$

Across the layer H is constant at any instant, but increases with t from 0 to $H_0 u_0/A_0$. On the other hand J increases with y from 0 to $(H_0 u_0/\eta)\exp(-A_0^2 t/\eta)$, the latter value decaying from $H_0 u_0/\eta$ to zero as t increases. The velocity profile at any instant is similar to RAYLEIGH'S [see (1)], the scale factor $\exp(-A_0^2 t/\eta)$ decreasing from 1 to 0.

★ Note that $\eta J = H_0 u$, so that $E = 0$ [see (5)].

Besides this similarity property of the velocity profiles there are two other interesting points to notice. First, the same result for u is obtained by assuming that the disturbance of the electro-magnetic field is small [$H = E = 0$ in equations (4) and (5)], but that the gradient of H, *i.e.* the current J, is important for the velocity field [$J = -\partial H/\partial y$ in (6) replaced by $-uH_0/\eta$ from (5)]. This is the argument used by Rossow [3] in discussing the present problem; what he in fact obtains is the boundary-layer equation for u, though under the unnecessary assumption that H/H_0 is small. However, this does not give H, which can only be obtained by determining the flow outside the boundary layer. His calculation of the total shearing stress on a pole-piece (*cf.* Section 9) is wrong since it involves using this boundary-layer approximation outside the layer. Secondly, the assumption that the magnetic field is not disturbed at the pole-piece is incorrect in $O(1/\eta^{\frac{1}{2}})$ for η large.

Owing to the presence of the exponential factor in u, the boundary layer effectively disappears in a time of the order of η/A_0^2. For a poor conductor this time may be quite long, and there will be a boundary layer of the same type as Rayleigh's ($\eta = \infty$) at the pole-piece. On the other hand, for small η, the boundary layer will barely form, and the initial discontinuity may be expected to be propagated away as a diffused Alfvén wave*. This we now show by discussing the inviscid flow outside the boundary layer.

This time we keep both η and y fixed as we let $\nu \to 0$, and obtain

$$H = \frac{H_0 u_0}{A_0} \cdot \frac{e^{-ny}}{p\sqrt{1+\eta p/A_0^2}}, \qquad u = \frac{u_0}{p}\left[1 - \frac{1}{1+\eta p/A_0^2} e^{-ny}\right],$$

$$J = \frac{H_0 u_0}{A_0^2} \cdot \frac{e^{-ny}}{1+\eta p/A_0^2},$$

where n is given in (20). It is easily checked that this is in fact the solution of the system (8), for $\nu = 0$, which satisfies the boundary condition $E = 0$ (but not $dH/dy = 0$) on $y = 0$. Moreover the values of H, u, and J for $y = 0$ agree with the corresponding values in (21) and (22) for $y = \infty$.

As in the case of general η, ν the inversion cannot be performed in terms of known functions. However, as there, the behavior of the inverses for small η can be determined by the method of steepest descent. The change of variable in the Bromwich integral is now $p = A_0^2 s/\eta$ and the large parameter becomes $\vartheta = A_0^2/\eta$. We omit the details, since they are now quite similar to those in Section 4, and only state the approximate result:

$$J \sim \frac{H_0 u_0}{A_0} \cdot \frac{1}{\sqrt{2\pi \eta t}} \exp\left\{-\frac{(y - A_0 t)^2}{2\eta t}\right\},$$

which is the same as that obtained from (19) by formally setting $\nu = 0$. The effective diffusion coefficient is $\frac{1}{2}\eta$.

6. Small and Large Values of *t*. The Residual Field

For unequal values of η and ν the flow must be computed from series expansions, if more than its asymptotic character for small η and ν is required. In obtaining these expansions we follow the ideas of Goldstein [2]. Thus for

* When the vanishing of η keeps pace with that of ν, as in Section 3, nothing that can be identified as a boundary layer even forms.

small t, we treat p as large in equation (16) and expand in terms of $P = p + a/b$:

$$\exp\left[-\sqrt{a+bp} \pm \sqrt{a+cp}\right] y$$
$$= \exp\left[-(\sqrt{b} \mp \sqrt{c})\sqrt{P}\,y\right] \cdot \exp \pm \left[\frac{a(b-c)}{2b\sqrt{c}} \cdot \frac{1}{\sqrt{P}} - \frac{a^2(b-c)^2}{8b^2 c^{\frac{3}{2}}} \cdot \frac{1}{P^{\frac{3}{2}}} + \cdots\right] y$$
$$= \exp\left[-(\sqrt{b} \mp \sqrt{c})\sqrt{P}\,y\right]\left\{1 \pm \frac{a(b-c)}{2b\sqrt{c}}\,\frac{y}{\sqrt{P}} + \frac{a^2(b-c)^2}{8b^2 c}\,\frac{y^2}{P} \pm \cdots\right\},$$

$$\frac{\exp\left[-\sqrt{a+bp} \pm \sqrt{a+cp}\right] y}{\sqrt{(a+bp)(a+cp)}} = \frac{\exp\left[-(\sqrt{b}\mp\sqrt{c})\sqrt{P}\,y\right]}{\sqrt{bc}\,P} \times$$
$$\times \left\{1 \pm \frac{a(b-c)}{2b\sqrt{c}}\,\frac{y}{\sqrt{P}} + \frac{a(b-c)}{2bc}\left[\frac{a(b-c)}{4b}\,y^2 - 1\right]\frac{1}{P} \pm \cdots\right\}.$$

Hence we find that, when $\eta \neq \nu$,

$$J = \sqrt{\frac{2}{\pi}} \cdot \frac{H_0 u_0}{\eta - \nu} \cdot \exp\left[-\frac{A_0^2 t}{(\sqrt{\eta}+\sqrt{\nu})^2}\right] \times$$
$$(23) \qquad \times \left\{e^{-y^2/8\eta t}\left[D_{-1}\left(\frac{y}{\sqrt{2\eta t}}\right) + A(2t)^{\frac{1}{2}} D_{-2}\left(\frac{y}{\sqrt{2\eta t}}\right) + B(2t)\,D_{-3}\left(\frac{y}{\sqrt{2\eta t}}\right) + \cdots\right] - \right.$$
$$\left. - e^{-y^2/8\nu t}\left[D_{-1}\left(\frac{y}{\sqrt{2\nu t}}\right) - A(2t)^{\frac{1}{2}} D_{-2}\left(\frac{y}{\sqrt{2\nu t}}\right) + B(2t)\,D_{-3}\left(\frac{y}{\sqrt{2\nu t}}\right) - \cdots\right]\right\},$$

where
$$A = \frac{A_0^2\,y}{(\sqrt{\eta}-\sqrt{\nu})(\sqrt{\eta}+\sqrt{\nu})^2}, \qquad B = \frac{2A_0^2\sqrt{\eta\nu}}{(\eta-\nu)^2}\left[\frac{A_0^2\,y^2}{4\sqrt{\eta\nu}(\sqrt{\eta}+\sqrt{\nu})^2} - 1\right],$$

and D_n is the parabolic cylinder function of order n. By examining the convergence of the two expansions involved we see that for computational purposes $A_0^2\sqrt{\eta\nu}\,t/(\eta-\nu)^2$ must be small, while at the same time $(\sqrt{\eta}-\sqrt{\nu})\,y/\sqrt{\eta\nu t}$ should not be large.

The corresponding result for $\nu = 0$, *i.e.* for flow outside the boundary layer, can be obtained in the same way, or as the limit of the above (y fixed):

$$J = \sqrt{\frac{2}{\pi}} \cdot \frac{H_0 u_0}{\eta} \exp -\left[\frac{A_0^2 t}{\eta} + \frac{y^2}{8\eta t}\right] \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{2A_0^4\,y^2\,t}{\eta^3}\right)^{\frac{n}{2}} D_{-n-1}\left(\frac{y}{\sqrt{2\eta t}}\right).$$

This expansion can be used for computation whenever $A_0^2 y\sqrt{t}/\eta^{\frac{3}{2}}$ is small.

Each of the functions $H - H_0 u_0/A_0 p$, u, and J has an infinity of order one half at $p = -a/b$ in the p-plane, and no other singularity (the one at $p = -a/c$ is removable). Hence each behaves like $t^{-\frac{1}{2}} \exp\left[-A_0^2 t/(\sqrt{\eta}+\sqrt{\nu})^2\right]$ as $t \to \infty$ with y fixed. More particularly we find

$$(24a) \quad H \sim \frac{H_0 u_0}{A_0}\left\{1 - \frac{\sqrt{\eta}+\sqrt{\nu}}{\sqrt{\pi}\,A_0}\cosh\left[\frac{A_0\,y}{\sqrt[4]{\eta\nu}(\sqrt{\eta}+\sqrt{\nu})}\right] \cdot \frac{1}{\sqrt{t}} \cdot \exp\left[-\frac{A_0^2 t}{(\sqrt{\eta}+\sqrt{\nu})^2}\right]\right\},$$

$$(24b) \quad \frac{u}{u_0\sqrt{\eta}(\sqrt{\eta}+\sqrt{\nu})} \sim \frac{J}{H_0 u_0} \sim$$
$$\sim \frac{1}{\sqrt{\pi}\,A_0\sqrt[4]{\eta\nu}}\sinh\left[\frac{A_0\,y}{\sqrt[4]{\eta\nu}(\sqrt{\eta}+\sqrt{\nu})}\right] \cdot \frac{1}{\sqrt{t}} \cdot \exp\left[-\frac{A_0^2 t}{(\sqrt{\eta}+\sqrt{\nu})^2}\right].$$

The most important thing to notice is that there is a residual field $H = H_0 u_0/A_0$ and hence a residual shearing stress on the pole-piece. We shall return to this point in Sections 8 and 9.

7. Finite Separation of the Pole-pieces

If in the previous analysis the pole-piece is located at $y=y_0$ instead of at $y=0$, then y must be everywhere replaced by $y-y_0$. If in addition the fluid lies below the pole-piece ($y<y_0$) instead of above, then y must be replaced by y_0-y and the sign of H changed. We shall now show that, with finite separation h of the pole-pieces, the flow is the superposition of a series of flows, each of which is of one of these two kinds.

Consider, for example, J as given by equation (15). Since

$$\frac{\cosh m(y-\frac{1}{2}h)}{\cosh \frac{1}{2}m h} = \frac{e^{m(y-\frac{1}{2}h)}+e^{-m(y-\frac{1}{2}h)}}{e^{\frac{1}{2}mh}+e^{-\frac{1}{2}mh}} = \frac{e^{m(y-h)}+e^{-my}}{1+e^{-mh}}$$
$$= (e^{m(y-h)}+e^{-my})\sum_{j=0}^{\infty}(-1)^j e^{-mjh}$$
$$= \sum_{j=0}^{\infty}(-1)^j e^{-m(\overline{j+1}\,h-y)} + \sum_{j=0}^{\infty}(-1)^j e^{-m(y+jh)},$$

and similarly for $\cosh n(y-\frac{1}{2}h)/\cosh\frac{1}{2}nh$, it follows that

$$J=\sum_{j=0}^{\infty}(-1)^j \mathscr{J}^-(\overline{j+1}\,h)+\sum_{j=0}^{\infty}(-1)^j\mathscr{J}^+(-jh); \tag{25}$$

here $\mathscr{J}^+(y_0)$ and $\mathscr{J}^-(y_0)$ are the current functions of the two flows referred to above, respectively. Similar results are found for H and u.

The same conclusions would have been reached by the method of images; $\mathscr{J}^-(\overline{j+2}\,h)$ is the reflection, in $y=h$, of $-\mathscr{J}^+(-jh)$, which is itself the reflection of $\mathscr{J}^-(jh)$ in $y=0$.

The results obtained for infinite separation therefore yield immediate descriptions of the flow for finite separation. Thus, when $\eta=\nu$ (see Section 3), J is total effect in $0<y<h$ of diffusion sources $\pm H_0u_0/A_0$ moving with velocity A_0 away from the points $y=0, \pm 2h, \pm 4h, \ldots$ and $y=\pm h, \pm 3h, \ldots$, respectively, together with their negatives moving with velocity $-A_0$ away from the same points. For η and ν small and comparable (Section 4) the motion consists essentially of two diffused Alfvén waves, one starting at $y=0$ and suffering successive reflections at $y=h$ and $y=0$, and the other originating at $y=h$ and moving exactly oppositely to the first. In the limit of zero η and ν the velocity is always zero in the regions between the pole-pieces and the wave fronts, and alternately $\pm u_0$ between the fronts. On the other hand, H is always zero between the fronts, and is alternately $\pm H_0u_0/A_0$ and $\mp H_0u_0/A_0$ in the lower and upper of the aforementioned regions, respectively. The wave fronts themselves are moving current and vortex sheets.

As $\nu\to 0$ with η fixed, boundary layers form at the pole-pieces. The values of J and u in the lower boundary layer are again given by equations (21) and (22) of Section 5, since the correction terms for the presence of the second pole-piece are constructed so as to give zero contributions to J and u at the first. However, the correction terms for H do not sum to zero and consequently, although H is still constant throughout the boundary layer, its dependence on time is not given by (22) but by the function whose transform is

$$H=\frac{H_0u_0}{p\sqrt{\eta p+A_0^2}}\tanh\left(\frac{ph}{2\sqrt{\eta p+A_0^2}}\right).$$

For small t we obtain the H of (22), but for $t \to \infty$ this H tends to zero instead of $H_0 u_0 / A_0$. The corresponding formulas for the upper layer are obtained by replacing y with $h - y$ and changing the sign of H. Outside the boundary layers the motion can be described, at least for small η, as in the general case above.

8. The Initial and Final Stages of the Motion

Up to values of t comparable with the smallest of h^2/η, h^2/ν, and h/A_0, only the first few terms in each of the series in (25) will contribute to J for $0 < y < h$. In fact, for $0 < y < \frac{1}{2}h$, the first term of the second series will predominate, while, for $\frac{1}{2}h < y < h$, the same term in the first series will. Any term retained may then be evaluated by means of (23) if $\eta \neq \nu$, and (17) if $\eta = \nu$. Thus the initial stage of the motion is computed by correction of the solution for infinite separation.

For large values of t we apply the partial fraction rule. First note that $p = -a/b$ is now a removable singularity as well as $p = -a/c$. For if in equation (15), J is considered to be a function of $\sqrt{a + b p}$ and $\sqrt{a + c p}$, it is easily seen that it is an even function of each. Thus the only singularities are poles at the points determined by the vanishing of $\cosh \frac{1}{2} m h$ and $\cosh \frac{1}{2} n h$, *i.e.* by

$$m = (2j+1)\frac{\pi i}{h} \quad \text{and} \quad n = (2j+1)\frac{\pi i}{h},$$

where j is an integer. According to (12) these points are

$$p = p_j^{\pm} = -\frac{(2j+1)^2 \pi^2}{2h^2}\left[\eta + \nu \pm \sqrt{(\eta-\nu)^2 - \frac{4A_0^2 h^2}{(2j+1)^2 \pi^2}}\right]; \tag{26}$$

clearly only non-negative values of j need be considered.

In general these poles are simple, the residue at p_j^+ being

$$\frac{H_0 u_0}{4\eta\nu} \cdot \frac{1}{\sqrt{(a+b p_j^+)(a+c p_j^+)}} \cdot \frac{\cos[(2j+1)\pi(y-\frac{1}{2}h)/h]}{i(-1)^j[\frac{1}{2}h(dm/dp)_{p_j^+}]}$$
$$= -4H_0 u_0 \frac{\sin[(2j+1)\pi y/h]}{\sqrt{(2j+1)^2\pi^2(\eta-\nu)^2 - 4A_0^2 h^2}}$$

and that at p_j^- the negative of this value. Hence, on combining terms in pairs and assuming $h < \pi|\eta-\nu|/2A_0$, we find*

$$J = 8H_0 u_0 \sum_{j=0}^{\infty} \frac{\sin[(2j+1)\pi y/h]}{\sqrt{(2j+1)^2\pi^2(\eta-\nu)^2 - 4A_0^2 h^2}} e^{-\alpha_j t}\sinh\beta_j t, \tag{27}$$

where

$$\alpha_j = (2j+1)^2\frac{\pi^2(\eta+\nu)}{2h^2}, \qquad \beta_j = \frac{(2j+1)\pi}{2h^2}\sqrt{(2j+1)^2\pi^2(\eta-\nu)^2 - 4A_0^2 h^2}.$$

When $h \geqq \pi|\eta-\nu|/2A_0$, β_j is pure imaginary or zero for each value of j such that $2j+1 \leqq 2A_0 h/\pi|\eta-\nu|$ and the corresponding sinh-term in (27) changes into a sine-term or a t-term. Similarly, or by use of (8), we find

$$H = \frac{8h}{\pi} H_0 u_0 \sum_{j=0}^{\infty} \frac{\cos[(2j+1)\pi y/h]}{(2j+1)\sqrt{(2j+1)^2\pi^2(\eta-\nu)^2 - 4A_0^2 h^2}} e^{-\alpha_j t}\sinh\beta_j t,$$

$$u = 4u_0 \sum_{j=0}^{\infty} \sin[(2j+1)\pi y/h]\left\{\frac{(\eta-\nu)\sinh\beta_j t}{\sqrt{(2j+1)^2\pi^2(\eta-\nu)^2 - 4A_0^2 h^2}} + \frac{\cosh\beta_j t}{(2j+1)\pi}\right\} e^{-\alpha_j t}.$$

* Of course, separation of variables leads to the same result.

However, when $\eta \neq \nu$, the dominant term in these series for large t is not necessarily the first. For consider $p_j^{\pm}$, as defined in (26), to be a function of the continuous variable

$$\varkappa = \frac{(2j+1)^2\,\pi^2}{h^2}\,, \qquad 0 < \varkappa < \infty .$$

Then we find that as $\varkappa$ decreases, p_j^+ increases through real values to $-a(1/b+1/c)$, when $\varkappa = a(b-c)^2/bc$, and then becomes imaginary with real part increasing further to zero when $\varkappa = 0$. But p_j^- increases to a maximum of $-a/c$ for $\varkappa = a(b-c)/c$ before decreasing to $-a(1/b+1/c)$ at $\varkappa = a(b-c)^2/bc$, after which it takes values conjugate to p_j^+. So long as this maximum point cannot be exceeded for non-negative integer j, *i.e.* so long as

$$h \leqq \frac{\pi\,\sqrt[4]{\eta\,\nu}\,\left|\sqrt{\eta}-\sqrt{\nu}\right|}{A_0}\,,$$

the term $j=0$ dominates in (27). When

$$\frac{\pi\,\sqrt[4]{\eta\,\nu}\,\left|\sqrt{\eta}-\sqrt{\nu}\right|}{A_0} < h \leqq \frac{\pi\,|\eta-\nu|}{2A_0}\,,$$

however, one or other of the terms whose j's straddle the value

$$\frac{1}{2}\left\{\frac{A_0\,h}{\pi\,\sqrt[4]{\eta\,\nu}\,\left|\sqrt{\eta}-\sqrt{\nu}\right|} - 1\right\}$$

is the most important. As h increases further this term will remain dominant at first, and then at some value before the range

(28) $$h > \frac{\pi\sqrt{\eta+\nu}\,\left|\sqrt{\eta}-\sqrt{\nu}\right|}{\sqrt{2}\,A_0}$$

it will give way to the first term once more (which is now oscillatory).

In particular we see that there is no residual field H, as there was with infinite separation (Section 6). In fact, assuming (28) holds, we have

$$\begin{aligned} H \sim H_0 &\frac{8h\,u_0}{\pi\sqrt{4A_0^2h^2-\pi^2(\eta-\nu)^2}} \cos\left(\frac{\pi\,y}{h}\right) e^{-\pi^2(\eta+\nu)t/2h^2} \sin\left(\frac{\pi\sqrt{4A_0^2h^2-\pi^2(\eta-\nu)^2}}{2h^2}\,t\right) \\ &= \frac{4}{\pi}\cdot\frac{H_0\,u_0}{A_0}\cdot\cos\left(\frac{\pi\,y}{h}\right) e^{-\pi^2(\eta+\nu)t/2h^2} \sin\left(\frac{\pi\,A_0}{h}\,t\right), \end{aligned}$$

when h is very large. This result should be compared with (24a).

9. Shearing Stress at a Pole-piece

From the right-hand side of (6) or directly from its definition as the sum of viscous and Maxwell stresses, the total shearing stress is seen to be

$$\tau = \mu\,H_0 H + \varrho_0\,\nu\,\partial u/\partial y .$$

Hence, according to equations (14), the transform of its value on $y=0$ is given by

$$\begin{aligned} \frac{\tau}{\varrho_0} &= \frac{u_0}{p^2}\cdot\frac{m\,n}{m+n}\left[A_0^2 + \nu(p+\eta\,m\,n)\right] \\ &= \frac{u_0\,A_0^2}{2\sqrt{\eta\,\nu}}\cdot\frac{1}{p\sqrt{a+b\,p}} + \frac{u_0}{2}\left(1+\sqrt{\frac{\nu}{\eta}}\right)\cdot\frac{1}{\sqrt{a+b\,p}}\,, \end{aligned}$$

of which the inverse is

$$\frac{\tau}{\varrho_0} = u_0 A_0 \operatorname{erf}\left(\frac{A_0 \sqrt{t}}{\sqrt{\eta}+\sqrt{\nu}}\right) + u_0 \sqrt{\frac{\nu}{\pi t}} \exp\left[-\frac{A_0^2 t}{(\sqrt{\eta}+\sqrt{\nu})^2}\right].$$

For small values of t this gives

$$\frac{\tau}{\varrho_0} = u_0 \sqrt{\frac{\nu}{\pi t}} + \frac{u_0 A_0^2 (2\sqrt{\eta}+\sqrt{\nu})}{\sqrt{\pi}(\sqrt{\eta}+\sqrt{\nu})^2} \sqrt{t} + O(t^{\frac{3}{2}}), \tag{29}$$

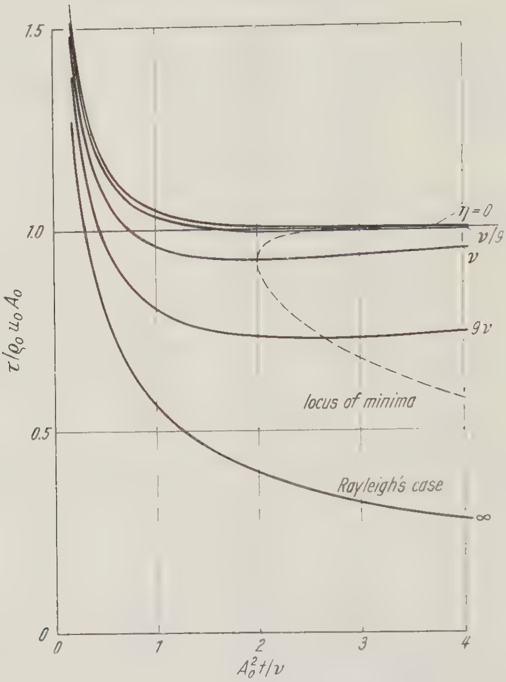

Fig. 4. The total stress τ on a pole-piece as a function of time, for various values of η

while for large t we find

$$\frac{\tau}{\varrho_0} \sim u_0 A_0 - \exp\left[-\frac{A_0^2 t}{(\sqrt{\eta}+\sqrt{\nu})^2}\right] \cdot \left[u_0 \sqrt{\frac{\eta}{\pi t}} + O(t^{-\frac{3}{2}})\right], \tag{30}$$

in agreement with equations (24). The first term in (29) is the value (2) found by RAYLEIGH [*1*] for the purely viscous case ($\eta = \infty$); the first term in (30) is the residual Maxwell stress arising from the residual magnetic field (see end of Section 6). Hence τ decreases from infinity more slowly* than in RAYLEIGH'S case, reaches a minimum [at $t = \sqrt{\nu}(\sqrt{\eta}+\sqrt{\nu})^2/2A_0^2\sqrt{\eta}$], and then tends asymptotically to $\varrho_0 u_0 A_0$, see Fig. 4.

When the pole-pieces have finite separation the corresponding stress (at each of them) is given by

$$\frac{\tau}{\varrho_0} = \frac{m n}{m^2 - n^2} \cdot \frac{u_0}{p^2} \times$$
$$\times \left[-n\{A_0^2 + \nu(p - \eta m^2)\} \tanh \tfrac{1}{2} m h + m\{A_0^2 + \nu(p - \eta n^2)\} \tanh \tfrac{1}{2} n h\right].$$

* Note, however, that the viscous part of the stress is always less than it is for $\eta = \infty$ by a factor $\exp[-A_0^2 t/(\sqrt{\eta}+\sqrt{\nu})^2]$.

This cannot be inverted in terms of known functions. However for sufficiently small values of t, depending on h, the inverse will differ little from (29), see Section 8. On the other hand, for large values of t we apply the partial fraction rule as in the preceding section and obtain

$$\frac{\tau}{\varrho_0} \sim \frac{4u_0}{\pi h}\left[\frac{2h^2 A_0^2+\pi^2\nu(\eta-\nu)}{\sqrt{4A_0^2h^2-\pi^2(\eta-\nu)^2}}\sin\left(\frac{\pi\sqrt{4A_0^2h^2-\pi^2(\eta-\nu)^2}}{2h^2}t\right)+\right.$$
$$\left.+\pi\nu\cos\left(\frac{\pi\sqrt{4A_0^2h^2-\pi^2(\eta-\nu)^2}}{2h^2}t\right)\right]e^{-\pi^2(\eta+\nu)t/2h^2}$$
$$\doteq \frac{4}{\pi}u_0A_0\sin\left(\frac{\pi A_0}{h}t\right)e^{-\pi^2(\eta+\nu)t/2h^2},$$

where it has been assumed that h is very large. Thus we see that, however large h is, τ is eventually oscillatory, the period tending to infinity and the damping to zero as h tends to infinity*.

Thus, for h large, the assumption $h=\infty$ leads to incorrect estimates for both H and τ for sufficiently large values of t. In other words the limit operations are not interchangeable:

$$\lim_{t\to\infty}\lim_{h\to\infty} \neq \lim_{h\to\infty}\lim_{t\to\infty},$$

for any values of η and ν.

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References

[1] Rayleigh, Lord: On the motion of solid bodies through viscous liquid. Phil. Mag., Ser. VI **21**, 697—711 (1911). — Scientific Papers, Vol. 6, pp. 29—40. London: Cambridge University Press 1920.

[2] Goldstein, S.: Some two-dimensional diffusion problems with circular symmetry. Proc. London Math. Soc., Ser. II **34**, 51—88 (1932).

[3] Rossow, Vernon J.: On flow of electrically conducting fluids over a flat plate in the presence of a transverse magnetic field. NACA Tech. Note 3971 (1957).

[4] Stewartson, K.: The dispersion of a current on the surface of a highly conducting fluid. Proc. Cambridge Phil. Soc. **53**, 774—775 (1957).

[5] Mises, R. von: Mathematical Theory of Compressible Fluid Flow, pp. 219—221. New York: Academic Press 1958.

[6] Friedrichs, K. O., & H. Kranzer: Nonlinear wave motion. Contract No AT(30-1)-1480, Institute of Mathematical Sciences, New York University Rept. NYO-6486 (1958), pp. 46—50.

* In fact, for every j, $p_j^{\pm}$ tend to zero through complex conjugate values as $h\to\infty$.

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Fehlerabschätzung bei linearen Gleichungssystemen mit dem Brouwerschen Fixpunktsatz

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Mit Hilfe des Brouwerschen Fixpunktsatzes ([*1*], [*7*]) werden Fehlerabschätzungen für lineare Gleichungssysteme $Gu = r$ hergeleitet. Alle vorkommenden Größen seien reell. Die gegebene Gleichung wird in der Form $u = Tu$ geschrieben und eine Menge $\mathfrak{D}$ ermittelt, welche der Operator T in sich abbildet. Diese Menge enthält dann nach dem Brouwerschen Fixpunktsatz einen Fixpunkt $u^* = Tu^*$, d.h. eine Lösung u^* des gegebenen Gleichungssystems. $\mathfrak{D}$ besteht hier aus allen Vektoren u, für welche $x \leqq u \leqq y$ bei geeignet gewählten Vektoren x, y gilt.

Die sich auf diese einfache Weise ergebende allgemeine Einschließungsaussage für die Lösung u^* wird für mehrere Sonderfälle formuliert und außerdem in verschiedener Weise umgeformt. Dabei stellt es sich heraus, daß eine Reihe von bekannten, bisher getrennt bewiesenen (z.T. aber noch nicht speziell auf Gleichungssysteme angewendeten) Fehlerabschätzungen in dem allgemeinen Ergebnis enthalten sind. Dazu gehören Fehlerabschätzungen, welche sich mit dem sogenannten Fixpunktsatz für kontrahierende Abbildungen ([*14*], [*4*] S. 35ff.) beweisen lassen, und solche, die man aus [*13*] folgern kann, sowie die in [*10*] angegebenen Abschätzungen; ferner die bei monoton wachsendem oder fallendem Operator mögliche Einschließung der Lösung zwischen die Näherungen eines Iterationsverfahrens ([*6*], [*2*], [*8*], [*11*], [*5*]) und die Einschließungsaussagen bei Gleichungssystemen monotoner Art [*3*]. Verschiedene dieser Abschätzungen werden verbessert und verallgemeinert, die Klassen von Gleichungssystemen, für welche solche Abschätzungen gelten, erweitert. Es müßte noch untersucht werden, ob man auf diese Weise wesentlich neue Typen von Gleichungssystemen behandeln kann. — Die Ergebnisse sind in § 7 zusammengefaßt.

Die Resultate kann man ohne große Schwierigkeiten auf lineare Gleichungen in einem Banachschen Raum übertragen, indem man den Schauderschen Fixpunktsatz benutzt. Damit sind dann z.B. Anwendungen auf Differentialgleichungen möglich. Derartige Anwendungen sollen jedoch zusammen mit einer funktionalanalytischen Behandlung nichtlinearer Probleme erst in einer folgenden Arbeit besprochen werden.

§ 1. Bezeichnungen

$A, B, \ldots$ bedeuten m-reihige quadratische Matrizen mit den Elementen a_{ik}, $b_{ik}, \ldots$ und $r, \ldots, u, v, \ldots$ m-dimensionale Vektoren mit den Komponenten $r^i, \ldots, u^i, v^i, \ldots$.

In der Menge dieser Vektoren wird eine Beziehung $u \leqq v$ in folgender Weise definiert:

$$u \leqq v \quad \text{bedeutet} \quad \sum_{k=1}^{m} h_{ik} u^k \leqq \sum_{k=1}^{m} h_{ik} v^k \qquad (i = 1, 2, \ldots, m). \tag{1.1}$$

Dabei ist $H = (h_{ik})$ eine gegebene reguläre Matrix. Praktisch wichtige Beispiele sind:

1. $u \leqq v$ bedeutet $u^i \leqq v^i$ für alle $i = 1, 2, \ldots, m$; (1.2)
2. $u \leqq v$ bedeutet $u^i \leqq v^i$ für bestimmte Nummern $i = i_1, i_2, \ldots, i_l$, (1.3)

$$u^i \geqq v^i \quad \text{sonst.}$$

(Wenn die Aufteilung (2.2) so gewählt wird, daß die Ergebnisse von der Numerierung der Unbekannten unabhängig sind, kann man im Falle (1.3) etwa

$$i_j = j \qquad (j = 1, 2, \ldots, l) \tag{1.4}$$

annehmen.)

Jeder Vektor u läßt sich dann in eine Differenz

$$u = u^+ - u^- \tag{1.5}$$

von Vektoren u^+ und u^- mit folgenden Eigenschaften aufspalten:

a) Es ist $o \leqq u^+$ und $u \leqq u^+$ (d.h. $u^- \geqq o$). (1.6)

b) Aus $o \leqq v$, $u \leqq v$ folgt $u^+ \leqq v$. (1.7)

$|u|$ sei der Vektor

$$|u| = u^+ + u^-. \tag{1.8}$$

Zum Beispiel erhält man im Falle

(1.2): $\quad |u| = (|u^i|)$,

(1.3): $\quad |u|^i = \begin{cases} \;\;|u^i| & \text{für} \quad i = i_j \quad (j = 1, 2, \ldots, l) \\ -|u^i| & \text{sonst,} \end{cases}$

(1.1): $\quad |u| = H^{-1}\left(\left|\sum_k h_{ik} u^k\right|\right)$.

Von dieser Ordnungsdefinition für Vektoren ausgehend, definieren wir eine solche für Matrizen:

$$A \leqq B \quad \text{bedeutet} \quad A u \leqq B u \quad \text{für jeden Vektor} \quad u \geqq o.$$

Der Ungleichung $A \leqq B$ entspricht bei der Ordnungsdefinition

(1.2): $\quad a_{ik} \leqq b_{ik} \qquad$ für alle i, k;

(1.3): $\quad a_{ik} \leqq b_{ik} \qquad$ für $i = i_\alpha$, $k = i_\beta$ und $i \neq i_\alpha$, $k \neq i_\beta$ $\quad (\alpha, \beta = 1, 2, \ldots, l)$
$\qquad\quad a_{ik} \geqq b_{ik} \qquad$ sonst;

(1.1): $\quad \tilde{a}_{ik} \leqq \tilde{b}_{ik} \qquad$ für alle i, k

bei

$$\tilde{A} = (\tilde{a}_{ik}) = H A H^{-1}, \qquad \tilde{B} = (\tilde{b}_{ik}) = H B H^{-1}. \tag{1.9}$$

Formal genau so wie u^+, u^-, $|u|$ werden dann A^+, A^- und $|A|$ erklärt. Man hat in (1.5) bis (1.8) nur o, u, v durch O, A, B zu ersetzen. Zum Beispiel erhält man im Falle

$$(1.2):\quad a_{ik}^+ = \tfrac{1}{2}\left(|a_{ik}| + a_{ik}\right), \quad a_{ik}^- = \tfrac{1}{2}\left(|a_{ik}| - a_{ik}\right) \quad \text{für alle } i, k;$$

$$(1.3):\quad a_{ik}^+ = \tfrac{1}{2}\left(|a_{ik}| + a_{ik}\right), \quad a_{ik}^- = \tfrac{1}{2}\left(|a_{ik}| - a_{ik}\right)$$

$$\text{für } i = i_\alpha,\ k = i_\beta \ \text{ und } \ i \neq i_\alpha,\ k \neq i_\beta \qquad (\alpha, \beta = 1, 2, \ldots, l),$$

$$a_{ik}^+ = \tfrac{1}{2}\left(-|a_{ik}| + a_{ik}\right), \quad a_{ik}^- = \tfrac{1}{2}\left(-|a_{ik}| - a_{ik}\right) \quad \text{sonst};$$

$$(1.1):\quad A^+ = H^{-1}\widetilde{A}^{(+)}H, \quad A^- = H^{-1}\widetilde{A}^{(-)}H$$

mit

$$\tilde{a}_{ik}^{(+)} = \tfrac{1}{2}\left(|\tilde{a}_{ik}| + \tilde{a}_{ik}\right), \quad \tilde{a}_{ik}^{(-)} = \tfrac{1}{2}\left(|\tilde{a}_{ik}| - \tilde{a}_{ik}\right) \quad \text{für alle } i, k$$

und (1.9).

§ 2. Allgemeine Einschließungsaussagen und Fehlerabschätzungen für Iterationsverfahren

2.1. Herleitung der allgemeinen Einschließungsaussagen. Die Matrix G des gegebenen Gleichungssystems

$$G u = r \tag{2.1}$$

werde aufgespalten in eine Differenz

$$G = A - B \tag{2.2}$$

mit einer regulären Matrix A. Die Gleichung $u = Tu$ mit dem durch

$$Tu = Mu + s, \quad M = A^{-1}B, \quad s = A^{-1}r$$

definierten Operator ist dann dem Ausgangsproblem (2.1) äquivalent. Wir suchen also einen Fixpunkt der durch T vermittelten Abbildung.

$\mathfrak{D}$ sei die Menge der Vektoren u, für welche

$$x \leqq u \leqq y$$

gilt, wobei x und y zwei feste Vektoren mit $x \leqq y$ bedeuten. Für $u \in \mathfrak{D}$ schätzt man ab:

$$Tu = Mu + s = M^+u - M^-u + s \leqq M^+y - M^-x + s \tag{2.3}$$

und entsprechend

$$Tu \geqq M^+x - M^-y + s. \tag{2.4}$$

Unter der Voraussetzung

$$x - M^+x + M^-y \leqq s \leqq y - M^+y + M^-x$$

gilt also $x \leqq Tu \leqq y$ für $u \in \mathfrak{D}$, d.h. der stetige Operator T bildet die konvexe, abgeschlossene und beschränkte Menge $\mathfrak{D}$ in sich ab. Diese Menge enthält deshalb nach dem Brouwerschen Fixpunktsatz ([1], [7]) einen Fixpunkt $u^* = Tu^*$ der Abbildung T, d.h. eine Lösung u^* des Gleichungssystems (2.1). Wir fassen das Ergebnis zusammen:

Ergebnis:

Voraussetzung:

$$x - M^+x + M^-y \leqq s \leqq y - M^+y + M^-x, \qquad x \leqq y \tag{2.5}$$

oder

$$x \leqq Tx - M^-(y-x), \quad Ty + M^-(y-x) \leqq y, \quad x \leqq y \tag{2.5'}$$

oder

$$x \leqq Ty - M^+(y-x), \quad Tx + M^+(y-x) \leqq y, \quad x \leqq y. \tag{2.5''}$$

Fehlerabschätzung:

$$x \leqq M^+x - M^-y + s \leqq u^* \leqq M^+y - M^-x + s \leqq y \tag{2.6}$$

oder

$$x \leqq Tx - M^-(y-x) \leqq u^* \leqq Ty + M^-(y-x) \leqq y \tag{2.6'}$$

oder

$$x \leqq Ty - M^+(y-x) \leqq u^* \leqq Tx + M^+(y-x) \leqq y. \tag{2.6''}$$

Die genaueren der jeweils angegebenen Schranken folgen mit $u^* = Tu^*$ aus (2.3), (2.4). Gilt $A^{-1} \geqq O$, so kann man M^- und M^+ durch die größeren Matrizen $A^{-1}B^-$ bzw. $A^{-1}B^+$ ersetzen, wie man an den Formulierungen (2.5′), (2.6′) bzw. (2.5″), (2.6″) erkennt.

2.2. Spezialfall des monoton wachsenden Operators. Es sei $M^- = O$, also $M \geqq O$. Dann wächst der Operator T monoton (im schwachen Sinne), d.h. aus $u \leqq v$ folgt $Tu \leqq Tv$. Definiert man

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = Tx_n, \quad y_{n+1} = Ty_n \quad (n = 0, 1, 2, \ldots), \tag{2.7}$$

so geht (2.5′) in (2.10) und (2.6′) in (2.11) über. Unter der Voraussetzung (2.10) gilt ferner (2.12), denn sei

$$x_0 \leqq x_1 \leqq \cdots \leqq x_n \leqq u^* \leqq y_n \leqq \cdots \leqq y_1 \leqq y_0 \tag{2.8}$$

für $n = p$ ($\geqq 1$) — für $p = 1$ ist dies nach (2.11) richtig —, so folgt

$$Tx_0 \leqq Tx_1 \leqq \cdots \leqq Tx_p \leqq Tu^* \leqq Ty_p \leqq \cdots \leqq Ty_1 \leqq Ty_0,$$

d.h. (2.8) für $n = p + 1$ und damit alle n.

Ergebnis:

Voraussetzung:

$$M^- = O \quad (d.h.\ M \geqq O) \tag{2.9}$$

und

$$x_0 \leqq y_0, \quad x_0 \leqq x_1, \quad y_1 \leqq y_0. \tag{2.10}$$

Fehlerabschätzung:

$$x_0 \leqq x_1 \leqq u^* \leqq y_1 \leqq y_0 \tag{2.11}$$

und

$$x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq u^* \leqq \cdots \leqq y_2 \leqq y_1 \leqq y_0. \tag{2.12}$$

Iterationsverfahren mit monoton wachsendem Operator in halbgeordneten Räumen behandelten L. V. KANTOROVITCH [*6*] und D. MORGENSTERN [*8*]. Praktische Anwendungen des monotonen Verhaltens von Iterationsfolgen bei linearen Gleichungssystemen zur numerischen Einschließung der Lösung findet man bei L. COLLATZ [*2*]. In der obigen Formulierung stehen die Ergebnisse für Verfahren in halbgeordneten Räumen auch in [*11*] und [*5*].

2.3. Spezialfall des monoton fallenden Operators. Ist $M^+=O$ also $M\leqq O$, so fällt der Operator T monoton: aus $u\leqq v$ folgt $Tu\geqq Tv$. Mit der Bezeichnung (2.7) geht (2.5″) in (2.14) sowie (2.6″) in (2.15) über. Entsprechend wie bei monoton wachsendem Operator beweist man hier außerdem (2.16) für die durch (2.7) erklärten Folgen x_n und y_n.

Ergebnis:

Voraussetzung:

$$M^+=O \qquad (d.h.\ M\leqq O) \tag{2.13}$$

und

$$x_0\leqq y_0, \quad x_0\leqq y_1, \quad x_1\leqq y_0. \tag{2.14}$$

Fehlerabschätzung:

$$x_0\leqq y_1\leqq u^*\leqq x_1\leqq y_0 \tag{2.15}$$

und

$$x_0\leqq y_1\leqq x_2\leqq y_3\leqq\cdots\leqq u^*\leqq\cdots\leqq x_3\leqq y_2\leqq x_1\leqq y_0. \tag{2.16}$$

Im Spezialfall $y_0=x_1$ lautet die Voraussetzung (2.14)

$$x_0\leqq x_1, \quad x_0\leqq x_2 \tag{2.17}$$

und die Fehlerabschätzung (2.16):

$$x_0\leqq x_2\leqq x_4\leqq\cdots\leqq u^*\leqq\cdots\leqq x_5\leqq x_3\leqq x_1. \tag{2.18}$$

Ein entsprechendes Ergebnis kann man für den Fall $x_0=y_1$ formulieren.

Die speziellen Aussagen (2.17), (2.18) für das abstrakte Iterationsverfahren mit monoton fallendem Operator wurden in [*11*] formuliert. Dort wurde auch ihre Anwendung auf lineare Gleichungssysteme kurz angedeutet. Die allgemeinen Aussagen (2.14), (2.16) findet man in [*5*] funktionalanalytisch bewiesen.

2.4. Fehlerabschätzungen für eine Näherungslösung $u_1=Tu_0$. Es sei u_0 eine Näherungslösung für u^* und $u_1=Tu_0$. Wir setzen

$$x=u_0+v, \quad y=u_0+w$$

mit Vektoren v, w, für welche $v\leqq w$ gilt. Dann geht das Ergebnis von Nr. 1 über in

Ergebnis:

Voraussetzung:

$$v-M^+v+M^-w\leqq u_1-u_0\leqq w-M^+w+M^-v, \qquad v\leqq w. \tag{2.19}$$

Fehlerabschätzung:

$$v\leqq u^*-u_0\leqq w \tag{2.20}$$

und

$$M^+v-M^-w\leqq u^*-u_1\leqq M^+w-M^-v. \tag{2.21}$$

Im Spezialfall $v=-w$ erhält man als

Ergebnis:

Voraussetzung:

$$|u_1-u_0|\leqq w-|M|w, \qquad w\geqq o. \tag{2.22}$$

Fehlerabschätzung:

$$|u^* - u_0| \leqq w \tag{2.23}$$

und

$$|u^* - u_1| \leqq |M|\, w. \tag{2.24}$$

Im Falle $A^{-1} \geqq O$ kann man in diesen Ergebnissen M^-, M^+ und $|M|$ durch die größeren Matrizen $A^{-1}B^-$, $A^{-1}B^+$ bzw. $A^{-1}|B|$ ersetzen.

Praktisch interessant dürften auch noch die Fälle $v = o$ und $w = o$ sein, insbesondere dann, wenn $M^- = O$ bzw. $M^+ = O$, also z.B. $A^{-1} \geqq O$ und $B^- = O$ bzw. $B^+ = O$ ist.

Das zweite der hier formulierten Ergebnisse (2.22), (2.23), (2.24) kann man auch aus [*13*], § 1 folgern. Jedoch wurde dies bisher noch nirgends ausführlich dargestellt. Die Sätze aus [*13*] liefern die zusätzliche Aussage, daß die durch $u_{n+1} = T u_n$ $(n = 0, 1, 2, \ldots)$ definierte Folge u_n unter der Voraussetzung (2.22) gegen u^* konvergiert, und sie sind auch auf komplexwertige Gleichungssysteme anwendbar. — Die mit (2.22) aus (2.24) folgende gegenüber (2.24) gröbere Abschätzung

$$|u^* - u_1| \leqq w - |u_1 - u_0| \tag{2.25}$$

wurde für die Ordnungsdefinition (1.2) und Aufspaltungen der in den §§ 4 und 5 behandelten Arten in [*10*] angegeben, allerdings unter einer zusätzlichen Konvergenzvoraussetzung. Mit der Schranke $\sigma = w - |u_1 - u_0|$ erhält man im Falle $A^{-1} \geqq O$ die für (2.22) hinreichende Bedingung

$$A\,\sigma \geqq |B|\left(\sigma + |u_1 - u_0|\right). \tag{2.26}$$

Das Ergebnis (2.22), (2.23), (2.24) stellt gleichzeitig auch eine Verbesserung von Aussagen dar, welche man mit dem Fixpunktsatz für kontrahierende Abbildungen $\left([14], [4]\text{ S. 35ff.}\right)$ bei der Norm $\|u\| = \operatorname*{Max}_i \frac{|u^i|}{w^i}$ $(w^i > 0)$ beweisen kann. Vgl. hierzu die Nrn. 4.3 und 5.3.

Während im Spezialfall $v = -w$ der Betrag $|u^* - u_1|$ abgeschätzt wird, besteht bei dem ersten der in dieser Nr. formulierten Ergebnisse die Möglichkeit, $u^* - u_1$ nach oben und unten in Schranken verschiedener Art einzuschließen, z.B. also auch Aussagen wie $u^* - u_1 \geqq o$ zu erhalten.

§ 3. Vom Defekt ausgehende Fehlerabschätzungen

3.1. Allgemeine Einschließungsaussagen. Es sei $A^{-1} \geqq O$. In diesem Falle kann man zur Fehlerabschätzung von hinreichenden Bedingungen für die in den Ergebnissen des § 2 genannten Voraussetzungen ausgehen, muß dann aber auf die schärferen der jeweils angegebenen Abschätzungen verzichten. So folgt aus (3.2) wegen (3.1) die Voraussetzung (2.5) mit $A^{-1}B^+$ statt M^+ und $A^{-1}B^-$ statt M^-.

Ergebnis:

Voraussetzung:

$$A^{-1} \geqq O \tag{3.1}$$

und

$$(A - B^+)\,x + B^- y \leqq r \leqq (A - B^+)\,y + B^- x, \qquad x \leqq y \tag{3.2}$$

oder (*statt* (3.2))

$$G\,x + B^-(y - x) \leqq r \leqq G\,y - B^-(y - x), \tag{3.3}$$

$$x \leqq y. \tag{3.4}$$

Fehlerabschätzung:
$$x \leqq u^* \leqq y. \tag{3.5}$$

(3.3) kann man als Bedingung für die Defekte $d_x = -Gx + r$ und $d_y = -Gy + r$ deuten.

L. Collatz [3] definierte Aufgaben $Gu = r$ monotoner Art in halbgeordneten Räumen so, daß $Gx \leqq Gy$ bei sonst beliebigen Elementen x, y immer $x \leqq y$ zur Folge haben soll. Diese Forderung ist bei linearen Gleichungssystemen mit $G^{-1} \geqq O$ identisch. Gilt $G^{-1} \geqq O$, so folgt wegen $r = Gu^*$ aus $Gx \leqq r \leqq Gy$ (d.h. (3.3) mit $B^- = O$) die Abschätzung (3.5). Diese Möglichkeit der Fehlerabschätzung bei Systemen monotoner Art ist in dem obigen Ergebnis enthalten: Man setze $A = G$ und $B = O$. Dann ist auch $B^- = O$ und (3.4) folgt aus (3.3) wegen $G^{-1} = A^{-1} \geqq O$. — Im Ergebnis (3.3), (3.4), (3.5) braucht das gegebene Gleichungssystem auch für den Fall $B^- = O$ nicht von monotoner Art zu sein.

3.2. Fehlerabschätzung für eine Näherungslösung u_0. Es sei u_0 eine Näherungslösung, deren Fehler abgeschätzt werden soll und d der zu u_0 gehörende Defekt
$$d = -Gu_0 + r$$
der Gleichung (2.1). Dann folgt im Falle (3.1) aus (3.6) durch Multiplikation mit A^{-1} die Voraussetzung (2.19) mit $A^{-1}B^+$ und $A^{-1}B^-$ statt M^+ und M^-.

Ergebnis:

Voraussetzung:
$$A^{-1} \geqq O$$
und
$$(A - B^+)\,v + B^- w \leqq d \leqq (A - B^+)\,w + B^- v, \qquad v \leqq w \tag{3.6}$$
oder (*statt* (3.6))
$$Gv + B^-(w - v) \leqq d \leqq Gw - B^-(w - v), \qquad v \leqq w. \tag{3.6'}$$

Fehlerabschätzung:
$$v \leqq u^* - u_0 \leqq w. \tag{3.7}$$

Im wichtigen Spezialfall $v = -w$ hat man das

Ergebnis:

Voraussetzung:
$$A^{-1} \geqq O \tag{3.8}$$
und
$$|d| \leqq (A - |B|)\,w, \qquad w \geqq o. \tag{3.9}$$

Fehlerabschätzung:
$$|u^* - u_0| \leqq w. \tag{3.10}$$

§ 4. Aufspaltung wie beim Gesamtschrittverfahren

4.1. Aufspaltung von G. Die Hauptdiagonalelemente der Matrix seien positiv:
$$g_{ii} > 0 \qquad (i = 1, 2, \ldots, m). \tag{4.1}$$
Wir spalten die Matrix G auf in die Summe einer Diagonalmatrix D, einer oberen Dreieckmatrix $-C_1$ und einer unteren Dreieckmatrix $-C_2$ (C_1 und C_2 sollen

auch in der Hauptdiagonalen nur Nullen enthalten):

$$G=\begin{bmatrix} & & -C_1\\ & D & \\ -C_2 & & \end{bmatrix} = D - C \quad \text{mit} \quad C = C_1 + C_2,$$

und setzen dann

$$G = A - B \quad \text{mit} \quad A = D \quad \text{und} \quad B = C.$$

Man erhält $Tu = D^{-1}Cu + D^{-1}r$, das Iterationsverfahren $u_{n+1} = Tu_n$ $(n = 0, 1, 2, \ldots)$ ist das sogenannte Verfahren in Gesamtschritten.

4.2. Monotonie beim Gesamtschrittverfahren. $u \leqq v$ sei durch (1.2) oder (1.3), (1.4) definiert. Dann ist $A^{-1} \geqq O$ wegen (4.1). Wir geben vier Fälle von Vorzeichenverteilungen bei den g_{ik} an (allgemein war (4.1) vorausgesetzt), für die $B^- = O$ oder $B^+ = O$ und damit dann auch $M^- = O$ bzw. $M^+ = O$ ist, der Operator T also monoton wächst bzw. fällt:

1. $g_{ik} \leqq 0$ für alle i, k mit $i \neq k$: $\qquad G = \begin{bmatrix} & & -\\ & + & \\ - & & \end{bmatrix}$[1],

2. $g_{ik} \geqq 0$ für alle i, k: $\qquad G = [\;+\;]$,

3. $g_{ik} \leqq 0$ für $i, k = 1, 2, \ldots, l$ und $i, k = l+1, \ldots, m$ mit $i \neq k$

   $g_{ik} \geqq 0$ sonst: $\qquad G = \left[\begin{array}{cc|cc} & - & & + \\ - & & & \\ \hline & & + & - \\ + & & - & \end{array}\right]$,

4. $g_{ik} \geqq 0$ für $i, k = 1, 2, \ldots, l$ und $i, k = l+1, \ldots, m$

   $g_{ik} \leqq 0$ sonst: $\qquad G = \left[\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right]$.

$$\text{Im Falle} \left\{\begin{matrix}1\\2\\3\\4\end{matrix}\right\} \text{ gilt } \left\{\begin{matrix}B^- = M^- = O\\ B^+ = M^+ = O\\ B^- = M^- = O\\ B^+ = M^+ = O\end{matrix}\right\} \text{ bei der Definition } \left\{\begin{matrix}(1.2)\\(1.2)\\(1.3),\ (1.4)\\(1.3),\ (1.4)\end{matrix}\right\}.$$

Das gewöhnliche Differenzenverfahren für Randwertaufgaben mit der ebenen oder räumlichen Potentialgleichung führt oft auf Gleichungssysteme, bei denen gleichzeitig die Bedingungen der Fälle 1 und 4 erfüllt sind, wenn man die Unbekannten in geeigneter Weise numeriert. Die Vorzeichenverteilung 2 erhält man z.B. bei den Clapeyronschen Gleichungen.

4.3. Fehlerabschätzungen für Näherungslösungen u_0 und u_1. Es sei u_0 eine Näherungslösung und u_1 die daraus nach dem Gesamtschrittverfahren berechnete

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[1] Die Vorzeichen $+$, $-$ bedeuten, daß die an den betreffenden Stellen stehenden Elemente $\geqq 0$ bzw. $\leqq 0$ sind; nur für die Hauptdiagonalelemente ist $g_{ii} > 0$ vorausgesetzt.

erste Näherung. $u \leqq v$ sei durch (1.2) definiert. Aus (2.22), (2.23), (2.24) und (3.9), (3.10) erhält man dann das

Ergebnis:

Voraussetzung:

$$|u_1^i - u_0^i| \leqq w^i - \frac{1}{g_{ii}} \sum_{\substack{k=1\\k \neq i}}^{m}{}' |g_{ik}|\, w^k, \qquad w^i \geqq 0 \qquad (i = 1, 2, \ldots, m)$$

oder

$$|d^i| = \left| -\sum_{k=1}^{m} g_{ik} u_0^k + r^i \right| \leqq g_{ii} w^i - \sum_{\substack{k=1\\k \neq i}}^{m}{}' |g_{ik}|\, w^k, \qquad w^i \geqq 0 \qquad (i = 1, 2, \ldots, m).$$

Fehlerabschätzung:

$$|u^{*i} - u_0^i| \leqq w^i \qquad (i = 1, 2, \ldots, m)$$

und

$$|u^{*i} - u_1^i| \leqq \frac{1}{g_{ii}} \sum_{\substack{k=1\\k \neq i}}^{m}{}' |g_{ik}|\, w^k \qquad (i = 1, 2, \ldots, m).$$

Insbesondere liefert der einfache Ansatz

$$w^i = \varepsilon \qquad (i = 1, 2, \ldots, m) \tag{4.2}$$

die folgenden Aussagen.

Ergebnis:

Voraussetzung:

$$\mu_i \leqq 1 \quad \text{mit} \quad \mu_i = \frac{1}{g_{ii}} \sum_{\substack{k=1\\k \neq i}}^{m}{}' |g_{ik}| \qquad (i = 1, 2, \ldots, m)$$

und

$$\left.\begin{array}{c} \varepsilon = \operatorname*{Max}_i \dfrac{|u_1^i - u_0^i|}{1 - \mu_i} < \infty \\ \left(\dfrac{|u_1^i - u_0^i|}{1 - \mu_i} = 0 \text{ gesetzt für } |u_1^i - u_0^i| = 0 \text{ und } \mu_i = 1\right). \end{array}\right. \tag{4.3}$$

Fehlerabschätzung:

$$|u^{*i} - u_0^i| \leqq \varepsilon \qquad (i = 1, 2, \ldots, m)$$

und

$$|u^{*i} - u_1^i| \leqq \mu_i\, \varepsilon \qquad (i = 1, 2, \ldots, m). \tag{4.4}$$

Ersetzt man in (4.3) und (4.4) alle μ_i durch $\tilde{\mu} = \operatorname*{Max}_i \mu_i$ — dabei ist dann $\tilde{\mu} < 1$ vorauszusetzen —, so ergeben sich Fehlerschranken für die $|u^{*i} - u_1^i|$, welche auch mit dem Fixpunktsatz für kontrahierende Abbildungen zu beweisen sind. — Hier erhält man im Falle $\tilde{\mu} < 1$ im allgemeinen schärfere Schranken, und man kommt auch im Falle $\tilde{\mu} = 1$ zu Abschätzungen, wenn man $|u_1^i - u_0^i| = 0$ für $\mu_i = 1$ erreichen kann.

§ 5. Aufspaltung wie beim Einzelschrittverfahren

5.1. Aufspaltung von G. Wir fordern wieder (4.1) und setzen jetzt

$$G = A - B \quad \text{mit} \quad A = D - C_2, \qquad B = -C_1$$

und den Bezeichnungen der Nr. 4.1. Dann wird $Tu = (D - C_2)^{-1} C_1 u + (D - C_2)^{-1} r$, und das Iterationsverfahren $u_{n+1} = T u_n$ $(n = 0, 1, 2, \ldots)$ ist das sogenannte Verfahren in Einzelschritten für das System (2.1).

5.2. Monotonie beim Einzelschrittverfahren. $u \leqq v$ sei durch (1.2) oder (1.3) mit

$$i_j = 2j - 1 \qquad (j = 1, 2, \ldots) \tag{5.1}$$

erklärt. Wir nennen Voraussetzungen, aus denen $A^{-1} \geqq O$ folgt, und solche, unter denen $M^- = O$ oder $M^+ = O$ ist. Die Elemente der Matrix $A^{-1} = (f_{ik})$ lassen sich in geschlossener Form angeben. Es sei

$$(i, \alpha_1, \alpha_2, \ldots, \alpha_p, k) = \frac{g_{i\alpha_1} g_{\alpha_1\alpha_2} g_{\alpha_2\alpha_3} \cdots g_{\alpha_p k}}{g_{ii} g_{\alpha_1\alpha_1} g_{\alpha_2\alpha_2} \cdots g_{kk}} \qquad \left(= \frac{g_{ik}}{g_{ii} g_{kk}} \text{ für } p = 0\right)$$

und

$$[i, k]_p = \sum_{i > \alpha_1 > \alpha_2 > \cdots > k} (i, \alpha_1, \alpha_2, \ldots, \alpha_p, k),$$

wobei über alle ganzzahligen α_j mit $i > \alpha_1 > \alpha_2 > \cdots > k$ summiert wird. Dann ist

$$f_{ik} = \sum_{p=0}^{i-k-1} (-1)^{p+1} [i, k]_p \quad \text{für} \quad i > k, \qquad f_{ii} = \frac{1}{g_{ii}}, \qquad f_{ik} = 0 \quad \text{für} \quad i < k.$$

$A^{-1} \geqq O$ gilt daher unter folgenden Bedingungen: Bei der Definition

(1.2): $\quad f_{ik} \geqq 0 \quad$ für alle i, k,

(1.3), (5.1): $\quad (-1)^{i+k} f_{ik} \geqq 0 \quad$ für alle i, k.

Dafür reicht hin bei der Definition

(1.2): $\quad g_{ik} \leqq 0 \quad$ für $i > k$,

(1.3), (5.1): $\quad (-1)^{i+k} g_{ik} \leqq 0 \quad$ für $i > k$.

Für folgende vier Vorzeichenverteilungen wächst oder fällt T monoton im Sinne einer der Definitionen (1.2) und (1.3), (5.1)[1]:

1. $G = \begin{bmatrix} + & & - \\ & \ddots & \\ - & & + \end{bmatrix}$, $\qquad$ 2. $G = \begin{bmatrix} + & & + \\ & \ddots & \\ - & & + \end{bmatrix}$,

3. $G = \begin{bmatrix} + & + & - & + & - & \cdots \\ + & + & + & - & + & \\ - & + & + & + & - & \\ + & - & + & + & + & \\ - & + & - & + & + & \\ \vdots & & & & & \end{bmatrix}$, $\qquad$ 4. $G = \begin{bmatrix} + & - & + & - & + & \cdots \\ + & + & - & + & - & \\ - & + & + & - & + & \\ + & - & + & + & - & \\ - & + & - & + & + & \\ \vdots & & & & & \end{bmatrix}$.

Im Falle $\left\{\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}\right\}$ gilt $A^{-1} \geqq O$ und $\left\{\begin{matrix} B^- = M^- = O \\ B^+ = M^+ = O \\ B^- = M^- = O \\ B^+ = M^+ = O \end{matrix}\right\}$ bei der Definition $\left\{\begin{matrix} (1.2) \\ (1.2) \\ (1.3),\ (5.1) \\ (1.3),\ (5.1) \end{matrix}\right\}$.

5.3. Fehlerabschätzungen für Näherungslösungen u_0 und u_1. u_0 sei eine Näherungslösung und u_1 die daraus mit dem Einzelschrittverfahren berechnete erste Näherung. $u \leqq v$ sei durch (1.2) oder (1.3), (5.1) erklärt. Von (2.22), (2.23),

(2.24) und (3.9), (3.10) ausgehend erhält man das

Ergebnis:

Voraussetzung:

$$|u_1 - u_0| \leqq \left(E - |(D - C_2)^{-1} C_1|\right) w, \qquad w \geqq o. \tag{5.2}$$

Hinreichende Bedingung (*für* (5.2)):

$$(D - |C_2|)\,|u_1 - u_0| \leqq (D - |C|)\, w, \qquad w \geqq o \tag{5.3}$$

oder

$$(D - C_2)^{-1} \geqq O, \quad |d| = |-G u_0 + r| \leqq \left(D - C_2 - |C_1|\right) w, \quad w \geqq o. \tag{5.4}$$

Fehlerabschätzung:

$$|u^* - u_0| \leqq w$$

und

$$|u^* - u_1| \leqq |(D - C_2)^{-1} C_1|\, w. \tag{5.5}$$

In (5.2) und (5.5) kann man $|(D-C_2)^{-1}C_1|$ durch die größeren Matrizen $|(D-C_2)^{-1}|\,|C_1|$ oder $(D-|C_2|)^{-1}|C_1|$ ersetzen. Daß die Bedingung (5.3) hinreicht, ergibt sich durch Multiplikation mit $(D-|C_2|)^{-1} \geqq O$. (5.4) ist die Voraussetzung (3.9), welche wegen $A^{-1} = (D - C_2)^{-1} \geqq O$ (2.22), d.h. (5.2) zur Folge hat.

Der einfache Ansatz (4.2) führt hier bei der Definition (1.2) auf folgendes

Ergebnis:

Voraussetzung:

$$\nu_i \leqq 1 \qquad (i = 1, 2, \ldots, m)$$

mit

$$\nu = (\nu_i) = \left(D - |C_2|\right)^{-1} |C_1|\, e, \quad e = (e^i), \quad e^i = 1 \qquad (i = 1, 2, \ldots, m) \tag{5.6}$$

und

$$\varepsilon = \operatorname*{Max}_i \frac{|u_1^i - u_0^i|}{1 - \nu_i} < \infty$$

$$\left(\frac{|u_1^i - u_0^i|}{1 - \nu_i} = 0 \text{ gesetzt für } |u_1^i - u_0^i| = 0 \text{ und } \nu_i = 1\right). \tag{5.7}$$

Fehlerabschätzung:

$$|u^{*i} - u_0^i| \leqq \varepsilon \qquad (i = 1, 2, \ldots, m) \tag{5.8}$$

und

$$|u^{*i} - u_1^i| \leqq \nu_i\, \varepsilon \qquad (i = 1, 2, \ldots, m). \tag{5.9}$$

Statt des Vektors ν in (5.6) kann man auch die kleineren Vektoren

$$\nu = |(D - C_2)^{-1}|\,|C_1|\, e \quad \text{oder} \quad \nu = |(D - C_2)^{-1} C_1|\, e,$$

im Falle $(D - C_2)^{-1} \geqq O$ also z.B. $\nu = (D - C_2)^{-1} |C_1|\, e$ verwenden. — Das genannte Ergebnis gilt auch, wenn man die Zahlen ν_i durch die größeren (in 4.3 definierten) μ_i ersetzt.

Benutzt man den Vektor ν aus (5.6) und in (5.7) und (5.9) statt ν_i überall $\tilde{\nu} = \operatorname*{Max}_i \nu_i$ — wobei $\tilde{\nu} < 1$ vorauszusetzen ist — so erhält man entsprechend wie beim Gesamtschrittverfahren in (5.9) eine Fehlerabschätzung, welche sich auch mit dem Fixpunktsatz für kontrahierende Abbildungen beweisen läßt (s. auch [9]).

§ 6. Allgemeines über die Aufteilung der Matrix G

6.1. Notwendige Bedingungen. In der Voraussetzung (2.5) ist die Forderung

$$x - M^+ x + M^- y \leqq y - M^+ y + M^- x, \qquad x \leqq y,$$

d.h.

$$z \geqq |A^{-1} B|\, z \quad \text{mit} \quad z = y - x \geqq o$$

enthalten. Ebenso folgert man aus (3.2)

$$A z \geqq |B|\, z, \qquad z = y - x \geqq o.$$

Die gleichen Bedingungen mit (z.T. anderen) Vektoren $z \geqq o$ fordern die Voraussetzungen aller in § 2 bzw. § 3 genannten Ergebnisse. Man hat also als

Notwendige Bedingung: *Damit die Voraussetzungen der in § 2 bzw. § 3 genannten Ergebnisse erfüllbar sind, muß es einen Vektor z mit folgenden Eigenschaften geben:*

$$\text{in § 2:} \qquad z \geqq o, \quad z \geqq |A^{-1} B|\, z, \tag{6.1}$$

$$\text{in § 3:} \qquad z \geqq o, \quad A z \geqq |B|\, z. \tag{6.2}$$

Besitzt z.B. die Eigenwertaufgabe

$$|A^{-1} B|\, z = \lambda z \quad \text{bzw.} \quad |B|\, z = \lambda A z$$

einen Eigenvektor $z \geqq o$ zu einem Eigenwert λ mit $0 \leqq \lambda \leqq 1$, so genügt dieser der Forderung (6.1) bzw. (6.2).

6.2. Aufteilung für Abschätzungen nach § 2. In den Abschätzungen des § 2 treten Vektoren der Form $T u = A^{-1}(B u + r)$ auf. Will man diese Ergebnisse benutzen, sollten also Gleichungssysteme mit der Matrix A einfach zu lösen sein. Das ist z.B. möglich, wenn A eine Diagonalmatrix ist wie in § 4 oder eine Dreieckmatrix wie in § 5 oder aber von solchen Formen nicht sehr stark abweicht, etwa bei Aufteilungen der Art (siehe [*15*]):

$$G = \begin{bmatrix} A & & -B & \\ & A & & \\ & & A & \\ -B & & & A \\ & & & & \ddots \end{bmatrix} \quad \text{oder} \quad G = \begin{bmatrix} A & & -B & \\ & A & & \\ & & A & \\ A & & & A \\ & & & & \ddots \end{bmatrix}.$$

Schließlich kann man auch Gleichungssysteme mit Matrizen der Gestalt[2]

$$\begin{vmatrix} c & -1 & & & & & \\ -1 & c & -1 & & & & \\ & -1 & c & -1 & & & \\ & & -1 & c & -1 & & \\ & & & \cdots & & & \\ & & & & -1 & c & -1 \\ & & & & & -1 & c \end{vmatrix}, \qquad c = \text{const}$$

oft recht einfach lösen, insbesondere dann, wenn m eine Potenz von 2 ist (s. [*12*]).

[2] Nullen als Elemente sind nicht hingeschrieben.

6.3. Aufteilung bei Abschätzungen nach § 3. Bei den Abschätzungen des § 3 darf A kompliziertere Gestalt haben. Hier bestehen u. U. Möglichkeiten, Fehlerschranken für wesentlich neue Typen von Gleichungssystemen zu gewinnen, für welche man bisher noch keine Fehlerschranken kennt. Dies müßte noch ausführlicher untersucht werden.

In § 3 wird $A^{-1} \geqq O$ verlangt. Diese Forderung ist bei der Definition (1.2) z. B. erfüllt, wenn

$$a_{ik} \begin{cases} > 0 & \text{für } i = k \\ \leqq 0 & \text{für } i \neq k \end{cases} \quad \text{und} \quad \sum_{k=1}^{m} a_{ik} \begin{cases} \geqq 0 & \text{für alle } i \\ > 0 & \text{für mindestens ein } i \end{cases}$$

gilt und die Matrix A nicht zerfällt [3]. Durch Produktbildung kann man weitere Matrizen mit $A^{-1} \geqq O$ erzeugen, z. B.[2]

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 5 & -4 & 1 & & \\ -4 & 6 & -4 & 1 & \\ 1 & -4 & 6 & -4 & 1 \\ & 1 & -4 & 6 & -4 \\ & & 1 & -4 & 5 \end{bmatrix}. \qquad (6.3)$$

Bei der Ordnungsdefinition (1.3), (5.1) gilt $A^{-1} \geqq O$, wie man aus dem eben genannten Ergebnis für die Definition (1.2) folgert, z. B. für die Matrix[2]

$$A = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}.$$

Die notwendige Bedingung (6.2) kann man auch in der Form

$$z \geqq o, \qquad G z \geqq 2 B^- z$$

schreiben. Sie ist also um so leichter erfüllbar, je kleiner B^- ist, am günstigsten wäre in diesem Sinne eine Aufteilung (2.2) mit $B^- = O$.

Bei der Ordnungsdefinition (1.2) bedeutet dies also, daß man möglichst viele der positiven g_{ik} zu A schlagen soll. Zum Beispiel enthält die Matrix (6.3) positive Elemente auch außerhalb der Hauptdiagonalen. Dreieckmatrizen A mit $A^{-1} \geqq O$ können ebenfalls positive Elemente g_{ik} mit $i \neq k$ aufweisen (s. Beispiel 2).

§ 7. Zusammenfassung

Die in den §§ 2 und 3 formulierten Ergebnisse enthalten u. a. Fehlerabschätzungen folgender Art:

a) Abschätzung des Fehlers $u^* - u_1$ einer Näherung $u_1 = T u_0$, ausgehend von Schranken für die Änderung $u_1 - u_0$, also Fehlerabschätzungen für das Iterationsverfahren $u_{n+1} = T u_n$ $(n = 0, 1, 2, \ldots)$ (Nr. 2.4).

b) Einschließung der Lösung zwischen Näherungsfolgen bei Iterationsverfahren mit monoton wachsendem oder fallendem Operator, ausgehend von einer Schachtelung der ersten Näherungen (Nrn. 2.2 und 2.3).

c) Abschätzung des Fehlers $u^* - u_0$ einer Näherungslösung u_0, ausgehend von Schranken für den Defekt $d = -Gu_0 + r$ (Nr. 3.2).

Zur praktischen Anwendung der Ergebnisse auf ein gegebenes Gleichungssystem $Gu = r$ hat man zunächst G in $G = A - B$ mit regulärer Matrix A aufzuspalten und die Beziehung $u \leqq v$ zu definieren. Es ist dann $Tu = A^{-1}Bu + A^{-1}r$. Dabei ist auf folgendes zu achten:

Die Matrix B soll möglichst „klein" gegenüber A sein (als notwendige Bedingung ist in allen Voraussetzungen enthalten, daß es einen Vektor z mit (6.1) bzw. (6.2) gibt).

Bei Abschätzungen der Arten a) und b) soll A so einfach gebaut sein, daß man ein Gleichungssystem mit der Matrix A leicht lösen kann.

Bei Abschätzungen der Art b) wird $A^{-1}B \geqq O$ bzw. $A^{-1}B \leqq O$ gefordert. (Für die beim Gesamt- und Einzelschrittverfahren üblichen Aufteilungen sind in den Nrn. 4.2 und 5.2 verschiedene Fälle genannt, bei denen eine dieser Ungleichungen erfüllt ist.)

Bei Abschätzungen der Art c) muß $A^{-1} \geqq O$ gelten, A kann dabei jedoch von komplizierterer Bauart sein. Hier sollte $B \geqq O$, jedenfalls aber B^- möglichst klein sein.

In den Voraussetzungen der Ergebnisse treten Vektoren auf, welche bestimmte Ungleichungen erfüllen sollen. Es seien zwei (allerdings eng zusammenhängende) Möglichkeiten genannt, solche Vektoren zu berechnen:

1. Man benutzt die Relaxationsrechnung, ermittelt die gesuchten Vektoren also durch systematisches „Probieren". — Bei b) geht man, um Vektoren x_0, y_0 der gewünschten Art zu bekommen, etwa von einer Näherungslösung u_0 aus, bringt Änderungen an und beobachtet deren Wirkung. Bei a) und c) kann man z.B. mit $v = w = o$ beginnen.

2. Man macht für die gesuchten Vektoren einen Parameter enthaltenden Ansatz und bestimmt die Parameter so, daß die jeweiligen Forderungen erfüllt sind. Bei b) kann der Ansatz z.B. $x_0 = u_0 - \delta\bar{v}$, $y_0 = u_0 + \varepsilon\bar{w}$ lauten, mit einer Näherungslösung u_0, festen Vektoren $\bar{v}, \bar{w}$ und Parametern δ, ε. Bei a) und c) setzt man etwa an: $v = -\delta\bar{v}$, $w = \varepsilon\bar{w}$. Zum Beispiel kommt man oft schon mit den einfachen Vektoren $\bar{v} = \bar{w} = (e^i)$, $e^i = 1$ $(i = 1, 2, \ldots, m)$, zum Ziel. (Die mit diesen Vektoren folgenden Ergebnisse für Aufteilungen wie beim Gesamt- oder Einzelschrittverfahren sind in den Nrn. 4.3 und 5.3 formuliert.)

§ 8. Beispiele

Beispiele für das Einschließen der Lösung mit der Relaxationsmethode bei monoton wachsendem Operator findet man in [2] und [4] (S. 341 ff.). In [*10*] werden Fehlerabschätzungen für Iterationsverfahren mit Hilfe von (2.26), (2.25)

Tabelle 1

G			r	A			B		
1,82	−2	0	9	2	−2	0	0,18	0	0
−1	1,84	−1	9	−1	2	−1	0	0,16	0
0	−1	1,9	9	0	−1	2	0	0	0,1

durchgeführt. Hier ermitteln wir Fehlerschranken für ein Beispiel durch Einschließen des Defektes und für ein zweites Beispiel mit Hilfe der Formel (5.9).

Tabelle 2

u_0	d	$\overline{w}$	$G\overline{w}$	$w(\leqq)$
100,53	$-0{,}0046$	1	0,08795	0,096
86,98	0,0068	$0{,}5\cdot\sqrt{3}$	0,09349	0,083
50,52	$-0{,}0080$	0,5	0,08397	0,048

Tabelle 3

z	$\tilde{d}$	$\tilde{\overline{w}}(\leqq)$
0,05	$-0{,}000200$	0,0693
0,0443	0,006472	0,0601
0,03	0,005700	0,0347

8.1. Beispiel 1. Bei der Randwertaufgabe

$$\ddot{\varphi}(t) + 1{,}62\,(1-t^2)\,\varphi(t) + 81 = 0, \qquad \varphi(-1) = \varphi(1) = 0$$

führt das gewöhnliche Differenzenverfahren zur Schrittweite $h=\frac{1}{3}$ auf das Gleichungssystem $Gu=r$ mit $G=A-B$, wobei G, r, A und B in Tabelle 1 aufgeführt sind. Die Näherungslösung u_0 in Spalte 1 der Tabelle 2 ergibt den Defekt $d=-Gu_0+r$ in Spalte 2.

$u\leqq v$ sei durch (1.2) definiert. Dann ist $A^{-1}\geqq O$ und $B\geqq O$. Gilt also (vgl. (3.6′), (3.7))

$$-Gw\leqq d\leqq Gw \qquad \text{für einen Vektor } w\geqq o, \tag{8.1}$$

so hat man die Fehlerabschätzung

$$-w\leqq u^*-u_0\leqq w.$$

Wir machen den Ansatz $w=\varepsilon\overline{w}$ mit dem Vektor $\overline{w}$ in Spalte 3, für welchen $A\overline{w}=(2-\sqrt{3})\,\overline{w}$ ist $\left(\overline{w}^i=\cos\frac{i}{3}\,\frac{\pi}{2}\right)$. Man erhält als $G\overline{w}$ den Vektor in Spalte 4, und die Forderung (8.1) ist erfüllt mit $\varepsilon=0{,}096$. Als Fehlervektor ergibt sich also $w=\varepsilon\overline{w}$ in Spalte 5.

Die Näherungslösung u_0 stimmt mit der exakten Lösung u^* in den angegebenen Stellen überein. Das kommt in der Fehlerabschätzung nicht zum Ausdruck. Wir können jedoch bei diesem verhältnismäßig „instabilen" Gleichungssystem auch nicht erwarten, daß sich vom Defekt ausgehend wesentlich genauere Schranken herleiten lassen, denn man kann u_0 in den angegebenen Stellen ändern, ohne daß sich die Komponenten des Defektes dem Betrage nach vergrößern.

Zum Beispiel würde der Vektor $\tilde{u}_0=u_0-z$ mit z in Spalte 1 der Tabelle 3 den Defekt $\tilde{d}$ in Spalte 2 ergeben. Vom Defekt ausgehend könnte man also $\tilde{u}_0$ für eine bessere Näherungslösung als u_0 halten. Führt man für $\tilde{u}_0$ eine Fehlerabschätzung wie oben durch, so ergibt sich $\tilde{\varepsilon}=0{,}0693$ und der Vektor $\tilde{w}=\tilde{\varepsilon}\overline{w}$ in Spalte 3 der Tabelle 3. Der Fehler von $\tilde{u}_0$ — er ist $\approx z$ — ist tatsächlich von der Größenordnung der Schranke $\tilde{w}$.

Man sieht an diesem Beispiel, daß man aus dem Defekt nicht ohne weiteres auf die Güte einer Näherungslösung schließen kann, sondern dazu eine Fehlerabschätzung braucht.

8.2. Beispiel 2. L. COLLATZ berechnet in [4] (S. 367) folgende Randwertaufgabe mit dem Differenzenverfahren:

$$\Delta\Delta\varphi = \text{const} \qquad \text{in dem trapezförmigen Gebiet der Abbildung,}$$

$$\varphi = \frac{\partial\varphi}{\partial\alpha} = 0 \qquad \text{auf der langen Trapezseite } (\alpha = \text{innere Normale}),$$

$$\varphi = \Delta\varphi = 0 \qquad \text{auf den anderen Seiten des Trapezes.}$$

Bei dem in die Abbildung eingezeichneten Dreieckgitter erhält man das Gleichungssystem $Gu = r$ mit der Matrix G in Spalte 1 der Tabelle 4 und dem Vektor r in Spalte 2.

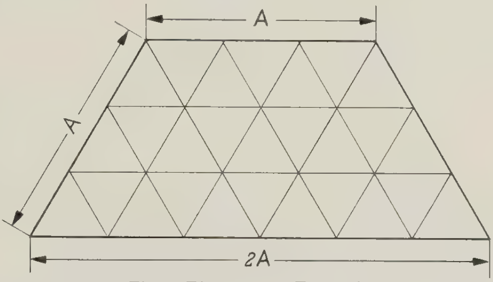

Fig. 1. Eingespannte Trapezplatte

$u \leqq v$ sei wieder durch (1.2) definiert. Zur Fehlerabschätzung benutzen wir das in Nr. 5.3 für den einfachen Ansatz (4.2) formulierte Ergebnis. Von der Näherungslösung u_0 in Spalte 3 ausgehend, berechnet man mit dem Einzelschrittverfahren die erste Näherung u_1 in Spalte 4 und die Änderung $u_1 - u_0$ in Spalte 5. Der Vektor v aus (5.6) steht in Spalte 6. Wir erhalten

$$\varepsilon = \frac{|u_1^1 - u_0^1|}{1 - v_1} = 0{,}001$$

und die Fehlerabschätzung

$$|u^{*i} - u_1^i| \leqq v_i\,\varepsilon \qquad (i = 1, 2, 3). \tag{8.2}$$

Der verhältnismäßig große Wert v_3 hat keinen Einfluß auf ε, da $|u_1^3 - u_0^3|$ sehr klein ist.

Man kann die Fehlerabschätzung etwas verbessern, indem man den aus

$$\begin{pmatrix} 12 & 0 & 0 \\ -3 & 8 & 0 \\ 2 & -12 & 11 \end{pmatrix} \hat{v} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}$$

zu berechnenden Vektor $\hat{v} = (D - C_2)^{-1} |C_1|$ statt v benutzt. Dies ist möglich, da $(D - C_2)^{-1} = (f_{ik}) \geqq O$ ist. Das einzige positive Element der Matrix $D - C_2$ außer-

Tabelle 4

G			r	u_0	u_1	$u_1 - u_0$	$v(\leqq)$	$\hat{v}(\leqq)$
12	−6	1	1	0,241	0,240583	−0,000417	0,584	0,584
−3	8	−3	1	0,394	0,394094	0,000094	0,594	0,594
2	−12	11	1	0,477	0,477087	0,000087	0,754	0,542

halb der Hauptdiagonalen, $g_{31} = 2$, hat nur Einfluß auf f_{31}, und eine kurze Rechnung liefert $f_{31} \geqq 0$. Mit dem Vektor $\hat{v}$ in Spalte 7 erhält man wieder $\hat{\varepsilon} = \varepsilon = 0{,}001$ und die Fehlerabschätzung (8.2) mit $\hat{v}_i$ statt v_i.

Literatur

[1] BROUWER, L. E. J.: Über die Abbildung von Mannigfaltigkeiten. Math. Ann. **71**, 97—115 (1912).
[2] COLLATZ, L.: Einschließungssätze bei Iteration und Relaxation. Z. angew. Math. Mech. **32**, 76—84 (1952).
[3] COLLATZ, L.: Aufgaben monotoner Art. Arch. Math. **3**, 366—376 (1952).
[4] COLLATZ, L.: Numerische Behandlung von Differentialgleichungen, 2. Aufl. Berlin-Göttingen-Heidelberg 1955.
[5] COLLATZ, L. u. J. SCHRÖDER: Einschließen der Lösungen von Randwertaufgaben. Numerische Mathematik **1** (1959).
[6] KANTOROVITCH, L. V.: The Method of Successive Approximation for Funktional Equations. Acta Math. **71**, 63—97 (1939).
[7] KNASTER, B., C. KURATOWSKI u. S. MAZURKIEWICZ: Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe. Fund. Math. **14**, 132—137 (1929).
[8] MORGENSTERN, D.: Beiträge zur nichtlinearen Funktionalanalysis. Diss. T. U. Berlin 1952.
[9] SASSENFALD, H.: Ein hinreichendes Konvergenzkriterium und eine Fehlerabschätzung für die Iteration in Einzelschritten bei linearen Gleichungen. Z. angew. Math. Mech. **31**, 92—94 (1951).
[10] SCHRÖDER, J.: Neue Fehlerabschätzungen für verschiedene Iterationsverfahren. Z. angew. Math. Mech. **36**, 168—181 (1956).
[11] SCHRÖDER, J.: Anwendung funktionalanalytischer Methoden zur numerischen Behandlung von Gleichungen. Z. angew. Math. Mech. **36**, 260—261 (1956).
[12] SCHRÖDER, J.: Über das Differenzenverfahren bei nichtlinearen Randwertaufgaben I. Z. angew. Math. Mech. **36**, 319—331 (1956).
[13] SCHRÖDER, J.: Über das Newtonsche Verfahren. Arch. Rat. Mech. Anal. **1**, 154—180 (1957).
[14] WEISSINGER, J.: Zur Theorie und Anwendung des Iterationsverfahrens. Math. Nachr. **8**, 193—212 (1952).
[15] WEISSINGER, J.: Verallgemeinerungen des Seidelschen Iterationsverfahrens. Z. angew. Math. Mech. **33**, 155—163 (1953).

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On the Inversion of Integral Transforms

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1. Introduction

This article deals with two methods for finding the inversion $F(x)$ of the integral transform

$$G(y) = \int_a^b K(y, x)\, F(x)\, d\,x \tag{1.1}$$

where $G(y)$ is a regular analytic function, defined in an open domain $\mathfrak{S}$ of the complex plane y.

These methods are the construction of the function $F(x)$ *(i)* from the values of the derivatives $G^{(j)}(y_0)$ of the function $G(y)$ at a point $y_0 \in \mathfrak{S}$, and *(ii)* from the values $G(y_j)$ of the function $G(y)$ at the points $y_j \in \mathfrak{S}$. Both methods have been described in the literature many times, usually in connection with the construction of $F(x)$ from a complete set of orthonormal functions [*1, 2, 3*]. It is the aim of this note to show that under certain conditions $F(x)$ can be constructed without any regard to the completeness of the system by which $F(x)$ is approximated.

The first method is based on the following consideration: Let the kernel $K(y, x)$ and an inversion $F(x)$ of the integral transform (1.1) be such that, for any $y_0 \in \mathfrak{S}$, the order of integration with respect to x and differentiation with respect to y can be interchanged. Then, the value of the derivative $G^{(j)}(y_0)$ is equal to the integral $\int_a^b K^{(j)}(y_0, x)\, F(x)\, d\,x$. This integral can be interpreted as a scalar product of the functions $F(x)$ and $q(x)\, \overline{K}^{(j)}(y_0, x)$, where $q(x)$ is a suitably selected weight function. Since the analytic function $G(y)$ is uniquely determined by the values of its derivatives $G^{(j)}(y_0)$, equation (1.1) may be replaced by the values of the scalar products of $F(x)$ with the $q(x)\, \overline{K}^{(j)}(y_0, x)$ and by the requirements that the transform $\int_a^b K(y, x)\, F(x)\, d\,x$ be regular and analytic for $y \in \mathfrak{S}$ and that the order of integration and of differentiation be interchangeable. Thus, it should be possible, with suitable choice of the weight function $q(x)$, to approximate in mean the function $F(x)$ by the set $\{q(x)\, \overline{K}^{(j)}(y_0, x)\}$.

For the second method we can say: Since $G(y)$ is a regular analytic function, it is determined by the values it takes at an infinite set of points $\{y_j\} \subset \mathfrak{S}$ having an accumulation point in $\mathfrak{S}$. The scalar product of the functions $F(x)$ and

$q(x)\bar{K}(y_j, x)$ is equal to $G(y_j)$; hence, equation (1.1) may be replaced by the scalar products of $F(x)$ with the $q(x)\bar{K}(y_j, x)$ and by the requirement that the transform $\int_a^b K(y, x) F(x)\, dx$ be regular analytic for $y \in \mathfrak{S}$. Thus we expect that we can approximate $F(x)$ by a set $\{q(x)\bar{K}(y_j, x)\}$.

2. Construction of $F(x)$ from derivatives of $G(y)$

A Hilbert space $L_q^2(a, b)$ is formed by all measurable complex-valued functions $f(x)$ having finite norms $\|f(x)\|$, where

$$\|f(x)\|^2 = \int_a^b |f(x)|^2 q^{-1}(x)\, dx \tag{2.1}$$

and the weight function $q(x)$ is measurable and positive almost everywhere. The scalar product (f, g) of two functions $f, g \in L_q^2(a, b)$ is defined by

$$\big(f(x), g(x)\big) = \int_a^b f(x)\bar{g}(x)\, q^{-1}(x)\, dx. \tag{2.2}$$

A function $F(x)$ satisfying (1.1) can be constructed from the derivatives $G^{(j)}(y_0)$ of the function $G(y)$ if we can find a weight function $q(x)$ such that the space $L_q^2(a, b)$ satisfies the following conditions:

1) At least one of the solutions of (1.1), say $F_1(x)$, belongs to $L_q^2(a, b)$.

2) For some $y_0 \in \mathfrak{S}$

$$q(x)\, K^{(j)}(y_0, x) \in L_q^2(a, b), \quad j = 0, 1, 2, \ldots. \tag{2.3}$$

3) For every $f(x) \in L_q^2(a, b)$

$$g(y) = \int_a^b K(y, x) f(x)\, dx \tag{2.4}$$

is a regular analytic function of y in the domain $\mathfrak{S}$.

4) For every $f(x) \in L_q^2(a, b)$

$$g^{(j)}(y_0) = \int_a^b K^{(j)}(y_0, x) f(x)\, dx, \quad j = 0, 1, 2, \ldots. \tag{2.5}$$

If the conditions 1) and 4) are satisfied, we can express the scalar products of $F_1(x)$ with the functions $q(x)\bar{K}^{(j)}(y_0, x)$ by the derivatives of $G(y)$, *viz.*,

$$G^{(j)}(y_0) = \big(F_1(x), q(x)\bar{K}^{(j)}(y_0, x)\big). \tag{2.6}$$

Now, from the set $\{q(x)\bar{K}^{(j)}(y_0, x)\}$, an orthonormal system of functions $\{\varphi_k(x)\}$ is formed with

$$\varphi_k(x) = q(x) \sum_0^{\varkappa}{}_j\, c_{kj} \bar{K}^{(j)}(y_0, x), \tag{2.7}$$

where $\varkappa \geqq k$, due to the possibility of linear dependence among the function $q(x)\bar{K}^{(j)}(y_0, x)$. The Fourier coefficients a_k of the function $F_1(x)$ can be expressed by the derivatives of $G(y)$,

$$a_k = \big(F_1(x), \varphi_k(x)\big) = \sum_0^{\varkappa}{}_j\, \bar{c}_{kj} G^{(j)}(y_0). \tag{2.8}$$

It follows from BESSEL's inequality that the series

$$F_{ap}(x) = \sum_{0}^{\infty}{}^{k}\, a_k \varphi_k(x) \tag{2.9}$$

converges in mean to some function $F_{ap}(x) \in L_q^2(a, b)$. By condition 3),

$$G_{ap}(y) = \int_a^b K(y, x)\, F_{ap}(x)\, dx \tag{2.10}$$

is a regular analytic function in the domain $\mathfrak{S}$. The Fourier coefficients of $F_1(x)$ and $F_{ap}(x)$ being equal, the derivatives of the functions $G(y)$ and $G_{ap}(y)$ are equal at the point y_0. Since both functions are regular analytic, we have for all $y \in \mathfrak{S}$

$$G(y) = G_{ap}(y), \tag{2.11}$$

and the function $F_{ap}(x)$ is a solution of (1.1).

The difference of the two solutions $F_1(x)$ and $F_{ap}(x)$ of (1.1),

$$F_0(x) = F_1(x) - F_{ap}(x), \tag{2.12}$$

is one of the solutions of the homogeneous equation

$$\int_a^b K(y, x)\, F_0(x)\, dx = 0. \tag{2.13}$$

Therefore we can conclude: The set $\{q(x)\, \overline{K}^{(j)}(y_0, x)\}$ is closed in $L_q^2(a, b)$ if, and only if, none of the non-trivial solutions of the homogeneous equation (2.13) belong to $L_q^2(a, b)$.

3. Construction of $F(x)$ from the values of the function $G(y)$

Let the Hilbert space $L_q^2(a, b)$, (2.1) and (2.2), satisfy the following conditions:

1) At least one solution, $F_1(x)$, of (1.1) belongs to $L_q^2(a, b)$.

2) It is possible to choose an infinite set of distinct points $\{y_j\} \subset \mathfrak{S}$ having an accumulation point in $\mathfrak{S}$ so that $\{q(x)\, \overline{K}(y_j, x)\} \subset L_q^2(a, b)$.

3) For every function $f(x) \in L_q^2(a, b)$,

$$g(y) = \int_a^b K(y, x)\, f(x)\, dx \tag{3.1}$$

is a regular analytic function of y in the domain $\mathfrak{S}$.

From the set $\{q(x)\, \overline{K}(y_j, x)\}$, the system of orthonormal functions $\{\varphi_k(x)\}$ is formed with

$$\varphi_k(x) = q(x) \sum_{0}^{\varkappa}{}^{j}\, c_{kj}\, \overline{K}(y_j, x), \qquad \varkappa \geqq k. \tag{3.2}$$

The Fourier coefficients a_k of the function $F_1(x)$ can be expressed as

$$a_k = \big(F_1(x), \varphi_k(x)\big) = \sum_{0}^{\varkappa}{}^{j}\, \bar{c}_{kj}\, G(y_j). \tag{3.3}$$

It follows from BESSEL's inequality that the series

$$F_{ap}(x) = \sum_{0}^{\infty}{}^{k}\, a_k \varphi_k(x) \tag{3.4}$$

converges in mean to some function $F_{ap}(x) \in L_q^2(a, b)$. By condition 3),

$$G_{ap}(y) = \int_a^b K(y, x)\, F_{ap}(x)\, dx \tag{3.5}$$

is a regular analytic function in the domain $\mathfrak{S}$. The Fourier coefficients of $F_1(x)$ and $F_{ap}(x)$ being equal, the regular analytic functions $G(y)$ and $G_{ap}(y)$ are equal in an infinite set $\{y_j\}$ of distinct points. Hence, for all $y \in \mathfrak{S}$

$$G_{ap}(y) = G(y), \tag{3.6}$$

and $F_{ap}(x)$ is also a solution of (1.1).

It is evident from the proof that it is not necessary that the condition 3) be satisfied by every $f(x) \in L_q^2(a, b)$. Strictly speaking, that it be satisfied at least by $F_1(x)$ and $F_{ap}(x)$ suffices. The same can be said for the third and the fourth conditions of the previous paragraph.

Besides, it follows that the set $\{q(x)\, \overline{K}(y_j, x)\}$ is closed in $L_q^2(a, b)$ if, and only if, none of the non-trivial solutions of the homogeneous equation (2.13) belong to $L_q^2(a, b)$.

4. Examples

a) The kernel $K(y, x)$ is a regular analytic function of $y \in \mathfrak{S}$ for every $a \leq x \leq b$, and a uniformly bounded function of $x \in [a, b]$ and $y \in \mathfrak{S}$. The difficulty in the application of both methods lies in the construction of the weight function $q(x)$ satisfying the conditions of Sections 2 and 3. The problem is simplified if the kernel $K(y, x)$ is a regular analytic function of $y \in \mathfrak{S}$ for $x \in [a, b]$ and a uniformly bounded function of $x \in [a, b]$ and $y \in \mathfrak{S}$, and if an inversion $F_1(x)$ exists which is integrable,

$$\int_a^b |F_1(x)|\, dx < M_0. \tag{4.1}$$

In this case we can always select $q(x)$ so that all conditions are satisfied for the application of both methods. We choose an almost everywhere positive measurable weight function so that

$$|F_1(x)|\, q^{-1}(x) < M_1 \tag{4.2}$$

and

$$\int_a^b q(x)\, dx < M_2, \tag{4.3}$$

where M_1 and M_2 are arbitrary. Let us verify that such a weight function satisfies all conditions for the inversion of the integral transform by both methods.

Clearly $F_1(x) \in L_q^2(a, b)$ since

$$\|F_1(x)\|^2 < M_1 M_0 < \infty. \tag{4.4}$$

The second condition of both methods is satisfied since, for every $y \in \mathfrak{S}$,

$$\|q(x)\, \overline{K}^{(j)}(y, x)\| < \sup_{a \leq x \leq b} |\overline{K}^{(j)}(y, x)|\, \sqrt{M_2} < \infty, \quad j = 0, 1, 2, \ldots \tag{4.5}$$

for the $K^{(j)}(y, x)$ are also bounded functions of $x \in [a, b]$ and $y \in \mathfrak{S}$, as one can see expressing $K^{(j)}(y, x)$ by $K(y, x)$ in the form of the Cauchy integral.

The third condition of both methods requires the function $g(y)$ to be regular and analytic in the domain $\mathfrak{S}$ for every $f(x) \in L_q^2(a, b)$. We shall show that $g(y)$ can be expanded in a power series arround every point $y_0 \in \mathfrak{S}$. Now,

$$K(y, x) = \sum_0^n{}_j \frac{(y-y_0)^j}{j!} K^{(j)}(y_0, x) + r_n(y, y_0, x), \tag{4.6}$$

and it follows from the expression of $r_n(y, y_0, x)$ as a contour integral involving $K(y, x)$ that, for each $y_0 \in \mathfrak{S}$, there exists an N that

$$|r_n(y, y_0, x)| < \varepsilon_0$$

for every

$$n > N, \quad |y - y_0| \leqq \varrho \vartheta_0 \quad \text{and} \quad x \in [a, b],$$

where $\varrho < 1$ and ϑ_0 is the distance between y_0 and the boundary $\mathfrak{S}$. Inserting the series (4.6) into the integral transform (2.4) we get

$$g(y) = \sum_0^n{}_j \frac{(y-y_0)^j}{j!} \left(f(x), q(x) \overline{K}^{(j)}(y_0, x)\right) + R_n(y, y_0),$$

where

$$R_n(y, y_0) = \left(f(x), q(x) \bar{r}_n(y, y_0, x)\right), \tag{4.7}$$

and

$$|R_n(y, y_0)| < \sqrt{M_2}\, \|f(x)\| \sup_{a \leqq x \leqq b} |r_n(y, y_0, x)| < \varepsilon \tag{4.8}$$

for every $n > N$ and $|y - y_0| \leqq \varrho \vartheta_0$. Thus $g(y)$ is indeed a regular analytic function of $y \in \mathfrak{S}$ for every $f(x) \in L_q^2(a, b)$.

It follows from equation (4.7), with $n = \infty$ and $R_\infty = 0$, that the fourth condition of Section 2 is also satisfied.

In the following two cases, the method will be applied to two kernels which are not analytic functions of $y \in \mathfrak{S}$ for every $x \in [a, b]$, but are uniformly bounded function of $x \in [a, b]$ and $y \in \mathfrak{S}$.

b) Inversion of Laplace transforms. Both methods can be applied to the inversion of Laplace transforms

$$g(y) = \int_0^\infty \exp(-yx) f(x)\, dx \tag{4.9}$$

provided y is in the domain

$$\mathfrak{S}_{\mathfrak{L}}; \quad \operatorname{Re} y > k, \quad |y| < M, \quad 0, k < M,$$

where M and k are arbitrary, and the weight function $q(x)$ is chosen so that

$$f(x) \in L_q^2(0, \infty)$$

and

$$q(x) \exp(-kx) \in L_q^2(0, \infty). \tag{4.10}$$

Consequently, the first two conditions of both methods are satisfied. By Schwarz' inequality we get from (4.10)

$$\int_0^\infty \exp(-kx)\, |f(x)|\, dx < \infty. \tag{4.11}$$

Thus, all other conditions are fulfilled too, as follows from the theory of Laplace transforms [4].

For an arbitrary solution of the integral equation (4.9) it is not always possible to choose a weight function $q(x)$ which satisfies (4.10), since solutions are known which do not satisfy the inequality (4.11) at any value of k [*4*].

If we put $q(x)=1$ and choose a positive y_0, the method of Section 2 becomes identical with Tricomi's method for the inversion of Laplace transforms by means of Laguerre polynomials [*1*].

With $q(x)=1$, the method of Section 3 becomes a particular case of the method of A. Erdélyi [*3*].

c) Inversion of Fourier transforms. The inversion of the Fourier transform

$$g(y)=\int_{-\infty}^{\infty}\exp(i\,y\,x)\,f(x)\,d\,x \tag{4.12}$$

is possible if we can choose the domain

$$\mathfrak{S}_{\mathfrak{F}};\qquad |\mathrm{Im}\,y|<k,\qquad |\mathrm{Re}\,y|<M\qquad k,M>0$$

and the weight function $q(x)$ so that

$$f(x)\in L_q^2(-\infty,\infty)$$

and

$$q(x)\exp(k|x|)\in L_q^2(-\infty,\infty). \tag{4.13}$$

That such a weight function satisfies all conditions of Section 2 and Section 3 can be shown in a similar way as in the case of the Laplace transforms.

Some types of integral equations cannot be solved by the described methods, *e.g.* integral equations of the second kind. However, the methods may be applicable to an equivalent integral equation obtained from the original one by a proper integral transform.

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References

[*1*] Tricomi, F.: Trasformazione di Laplace e polinomi di Laguerre. I., II. Atti Acad. naz. Lincei. Rend. Cl. Sci. fis. mat. nat. (6) **21** (1935).

[*2*] Morse, P. M., & H. Feshbach: Methods of Theoretical Physics, Part I, p. 931 to 935. New York: McGraw-Hill Book Co., Inc. 1953.

[*3*] Erdélyi, A.: Note on an Inversion Formula for the Laplace Transformation. J. London Math. Soc. **18**, 72–77 (1943).

[*4*] Doetsch, G.: Handbuch der Laplace-Transformation. Basel: Birkhäuser 1950.

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Vaguely Normal Operators on a Banach Space

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1. Introduction

This is the first of two articles concerned with extending to Banach space operators certain properties of normal (Hilbert space) operators.

If β is the Haar measure of a locally compact abelian group G, then the relation

$$(x|y) = \int_G x(\lambda) \cdot y(\lambda) \cdot \beta(d\lambda) \quad \text{for all } (x, y) \text{ in } L^p(G) \times L^{p'}(G)$$

defines (when $p=2$) the inner product in the space $L^2(G)$. If $\mathscr{R}(L^p(G))$ is the set of all operators T on $L^p(G) \cup L^{p'}(G)$ such that

$$(Tx|y) = (x|Ty) \quad \text{for all } (x, y) \text{ in } L^p(G) \times L^{p'}(G),$$

then $\mathscr{N}(L^p(G))$ will denote the set of all operators of the form $T_1 + i T_2$, where T_1 and T_2 are both in $\mathscr{R}(L^p(G))$. An operator will be called "vaguely normal" if it is a bounded operator on $L^p(G)$ that has an extension in $\mathscr{N}(L^p(G))$. I here take $G = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ and consider a certain algebra $\mathfrak{A}_{\otimes p}$ of convolution operators on $l_p = L^p(G)$. It turns out that $\mathfrak{A}_{\otimes p}$ consists of vaguely normal operators. Generalizing previous results [*5*, *6*], I show that $\mathfrak{A}_{\otimes p}$ is included in the weak closure of the algebra generated by the Hilbert transformation $x \to Hx = \{[Hx]_n\}_{n \in G}$, where

$$[Hx]_n = \sum_{\nu \in G} x_\nu \frac{1}{n - \nu} \quad (\nu \neq n). \tag{1}$$

To each T in $\mathfrak{A}_{\otimes p}$ corresponds on operator-valued measure E^T and a function f such that the relation

$$T = \int \lambda \cdot E^T(d\lambda) = \int_0^1 f(\vartheta) \cdot E^H(d\vartheta) \tag{2}$$

holds in a certain sense (see 5.5.). When $p=2$, this implies that $T = f(H)$ (using the functional notation customary in Hilbert space). If $p \geqq 2$ then the resolvent set of T is void (Theorem 5.4); this extends another property of normal operators [15, p. 305]. M. Riesz and E. C. Titchmarsh [*14*] have studied certain outstanding members of the algebra $\mathfrak{A}_{\otimes p}$ (see the examples in 5.7).

2. Three measure-theoretic lemmas.

Since this paragraph deals with immediate consequences of well known results in [*1*], the explanations will be very sketchy. It will be convenient to say that $(E, \mathfrak{S}, \nu)$ is a (TF) *complex measure space* if $\nu = \sum \omega_n \nu_n$ (where $n = 1, \ldots, 4$ and $\omega_n = \exp(i n \pi/2)$) and if $(E, \mathfrak{S}, \nu_n)$ is a totally finite measure space [*1*, p. 73 and p. 31] for all $n = 1, \ldots, 4$. Let μ be Lebesgue measure on $X = [0,1]$, while $\mathfrak{X}$ denotes the σ-ring of all μ-measurable subsets of X. All functions are complex-valued.

2.1 Lemma. Suppose that $h \in L^1(\mu, \mathfrak{X})$ and let ν be the function $E \to \nu(E)$ defined on $\mathfrak{X}$ by

$$\nu(E) = \int_E h \cdot d\mu;$$

then $(X, \mathfrak{X}, \nu)$ is a (TF) complex measure space, $\nu \ll \mu$, and $d\nu/d\mu = h$.

Note. By writing $h = \sum \omega_n h_n$, $(n = 1, \ldots, 4)$ with $h_n \geqq 0$ and $h_n \in L^1(\mu, \mathfrak{X})$, we can define $\nu_n(E) = \int_E h_n \, d\mu$; accordingly $\nu = \sum \omega_n \nu_n$. From [*1*, p. 127 (1)] follows that $\nu_n \ll \mu$ for all n (this is our definition of $\nu \ll \mu$). From [*1*, p. 133] follows that $d\nu_n/d\mu = h_n$, whence $h = \sum \omega_n d\nu_n/d\mu = d\nu/d\mu$; the last equality embodies our definition of $d\nu/d\mu$.

2.2 Lemma. Suppose that $(X, \mathfrak{X}, \nu)$ is a (TF) complex measure space and $\nu \ll \mu$. If g is a bounded Borel function in $L^1(\nu, \mathfrak{X})$, then

$$\int g \cdot d\nu = \int g \cdot \frac{d\nu}{d\mu} \cdot d\mu. \tag{3}$$

2.3 Note. Here $L^1(\nu, \mathfrak{X}) = \bigcap \{L^1(\nu_n, \mathfrak{X}) : n = 1, \ldots, 4\}$. Let g_0 and g_1 be the real and imaginary parts of g; the g_k are then bounded Borel functions and $g_k \in L^1(\nu, \mathfrak{X})$. By [*1*, p. 134, Theorem B], the equality (3) holds for $g = g_k$, $\nu = \nu_n$. A suitable linear combination of these results yields the conclusion, if we define

$$\int g \cdot d\nu = \sum_{k=0}^{1} \sum_{n=1}^{4} (i)^k \omega_n \int g_k \cdot d\nu_n.$$

2.4 Notation. If g and A are functions, then $(g \circ A)$ denotes the function $x \to g(A(x))$ (called gA by HALMOS [*1*, p. 162]). The set $L^\infty(\mu, \mathfrak{X})$ of all μ-measurable, essentially bounded functions will also be written $L^\infty(X)$.

2.5 Lemma. Suppose that $A \in L^\infty(\mu, \mathfrak{X})$, and let $\varpi(A)$ denote the closure of the range of A. If $\mathfrak{Y}$ denotes the class of all Borel sets of $\varpi(A)$, then the set $\mathfrak{S} = \{A^{-1}(F) : F \in \mathfrak{Y}\}$ is a σ-subring of $\mathfrak{X}$ [*1*, p. 162]. If g is a bounded Borel function defined on $\varpi(A)$ then $(g \circ A) \in L^1(\nu, \mathfrak{S})$ and

$$\int_X (g \circ A) \cdot d\nu = \int_{\varpi(A)} g \cdot d(\nu \circ A^{-1}).$$

Note that A is a measurable transformation (in the sense of [*1*, p. 162]). With the notation of 2.3, it follows from [*1*, p. 162, Theorem B] that all $(g_k \circ A)$ are $\mathfrak{S}$-measurable; but, since any bounded $\mathfrak{S}$-measurable function belongs to $L^1(\beta, \mathfrak{S})$ whenever $(X, \mathfrak{S}, \beta)$ is a totally finite measure space, it results that $(g_k \circ A) \in L^1(\nu_n, \mathfrak{S})$. Therefore $(g \circ A) \in L^1(\nu, \mathfrak{S})$. The rest of the proof follows from [*1*, p. 163] by means of the indications given in 2.3.

3. Laurent operators

Suppose that $\varphi \in L^{\infty}(X)$, $X=[0, 1]$, and let $Y\varphi$ denote the sequence $\{[Y\varphi]_n\}_{n\in G}$ of Fourier coefficients of φ;

$$[Y\varphi]_n = \int_0^1 e^{-2\pi i n\vartheta}\varphi(\vartheta)\,\mu(d\vartheta) \qquad (n\in G). \tag{4}$$

If $x=\{x_\nu\}_{\nu\in G}$, I shall denote by $\varphi\otimes x$ the convolution $[Y\varphi]*x$; in other words $\varphi\otimes x=\{[\varphi\otimes x]_n\}_{n\in G}$, where

$$[\varphi\otimes x]_n = \sum_{\nu\in G} x_\nu [Y\varphi]_{n-\nu}. \tag{5}$$

First studied by Toeplitz in 1907, the operators $\varphi_{\otimes 2}=\{b\in l_2\to\varphi_\otimes b\}$ were called "Laurent operators" by F. Riesz [*12*, pp. 171–176, 152–155]; they are represented by Toeplitz matrices (Hartman & Wintner [*4*]). It is assumed throughout that $1<p<\infty$. I shall write $p'=p/(p-1)$ and*

$$(x|y) = \sum_{n\in G} x_n\cdot\overline{y_n} \qquad ((x, y)\in l_p\times l_{p'}). \tag{6}$$

If $(b,c)\in l_2\times l_2$, there exists two functions B and C in $L^2(X)$ such that $b=[YB]$ and $c=[YC]$. Therefore $[Y\varphi]*b=[Y\varphi]*[YB]=[Y(\varphi\cdot B)]$ and accordingly

$$(\varphi\otimes b|c) = \int_0^1 \varphi(\vartheta)\,B(\vartheta)\,\overline{C(\vartheta)}\,\mu(d\vartheta) = \int \varphi\cdot B\cdot\overline{C}\cdot d\mu. \tag{7}$$

The first equality comes from the Plancherel theorem [7, 36D and 38B], the purpose of the second equality is to correlate with the notation (due to Halmos) which was used in § 2. It is easily seen that $\varphi_{\otimes 2}$ is a bounded operator on l_2.

3.1 Definition. *Let $\chi(E)$ be the characteristic function of a set E; then M^A will denote the mapping $\sigma\to M^A(\sigma)=\chi(A^{-1}(\sigma))_{\otimes 2}$ of the Borel sets of the plane into the space of bounded operators on l_2. On the other hand, $M(E)=\chi(E)_{\otimes 2}$ for all E in $\mathfrak{X}$* (see § 2).

3.2 Theorem. *Suppose that $A\in L^{\infty}(X)$. If g is a bounded Borel function defined on $\varpi(A)$* (see 2.5), *then the relation*

$$((g\circ A)\otimes b|c) = \int_0^1 g(A(\vartheta))\,(M(d\vartheta)\,b|c) = \int g(\lambda)\,(M^A(d\lambda)\,b|c)$$

holds when $(b, c)\in l_2\times l_2$.

Proof. Fix b and c, and write $\nu(E)=(M(E)\,b|c)$. A successive application of 3.1 and (7) yields

$$\nu(E) = (\chi(E)\otimes b|c) = \int \chi(E)\cdot B\cdot\overline{C}\,d\mu = \int_E (B\cdot\overline{C})\cdot d\mu.$$

Since $(B\cdot\overline{C})\in L^1(X)$, we can infer from 2.1 that $(X, \mathfrak{X}, \nu)$ is a (TF) complex measure space, $\nu\ll\mu$ and

$$\frac{d\nu}{d\mu} = B\cdot\overline{C} \qquad (\mu\text{-almost everywhere}). \tag{8}$$

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* The complex conjugate of λ is denoted $\overline{\lambda}$, and $\overline{C}$ is the function that satisfies $\overline{C}(\vartheta)=\overline{C(\vartheta)}$.

But, by 2.5, $f=(g\circ A)$ is a bounded function in $L^1(\nu, \mathfrak{S})$, and

$$\int f\cdot d\nu = \int f\cdot \frac{d\nu}{d\mu}\cdot d\mu = \int_0^1 f\cdot B\cdot \overline{C}\cdot d\mu = (f\otimes b|c). \tag{9}$$

The first equality comes from 2.2 (3), the second equality from (8), and the third comes from (7). On the other hand,

$$\int f\cdot d\nu = \int_0^1 f(\vartheta)\,\big(M(d\vartheta)\,b|c\big) = \int g(\lambda)\,\nu\big(A^{-1}(d\lambda)\big). \tag{10}$$

The first equality results from the definition of ν, while the second comes from 2.5. Since $\nu\big(A^{-1}(\sigma)\big)=\big(\chi(A^{-1}(\sigma))\,b|c\big)=\big(M^A(\sigma)\,b|c\big)$, a comparison of (9) and (10) concludes the proof.

3.3 Remarks. If $A\in L^\infty(X)$ it is easily checked that M^A is a compact complex spectral measure in the sense described in Halmos' book on Hilbert space [2]. Incidentally, the conventions used in [2] permit the more concise notation

$$A_{\otimes 2} = \int_0^1 A(\vartheta)\cdot M(d\vartheta) = \int \lambda\cdot M^A(d\lambda)$$

in the case $g(\lambda)=\lambda$.

3.4 Corollary. If $A\in L^\infty(X)$, $b\in l_2$ and $m\in G$, then

$$[A\otimes b]_m = \int_0^1 A\cdot dN_m \quad \text{and} \quad N_m \ll \mu.$$

Proof. Substitute $c=\{\delta_{mk}\}_{k\in G}$ (Kronecker delta) in 3.2 with $g(\lambda)=\lambda$ and set $N_m(E)=\big(M(E)\,b|c\big)=[M(E)\,b]_m$.

4. The property S_p

Let $\mathfrak{E}_p$ be the class of all bounded operators which map l_p into itself. If Q is an operator defined on $l_p\cup l_{p'}$, then Q_r will denote its restriction to l_r (i.e., $Q_r x = Qx$ for all x in l_r); the class $\mathfrak{E}_p\cap\mathfrak{E}_{p'}$ consists of all such operators Q which satisfy $Q_p\in\mathfrak{E}_p$ and $Q_{p'}\in\mathfrak{E}_{p'}$.

If $A\in L^\infty(X)$, define $A'=\{\vartheta\to A(1-\vartheta)\}$ and consider the operations $A\to A'$ and $A\to\bar{A}$. From now on, μ-equivalent functions are identified. I shall denote by S_p the family of all subalgebras $\mathfrak{F}$ of $L^\infty(X)$ which are closed under these two operations and which are such that $A\to A_\otimes$ is an algebraic isomorphism of $\mathfrak{F}$ into a subalgebra $\mathfrak{F}_\otimes$ of $\mathfrak{E}_p\cap\mathfrak{E}_{p'}$.

4.1 Remark. It is easily checked that $L^\infty(X)\in S_2$.

4.2 Notation. If $Q=A_\otimes$ I write $Q'=(A')_\otimes$ and $\overline{Q}=(\bar{A})_\otimes$. If $(x,y)\in l_p\times l_{p'}$ then $\langle x,y\rangle=(x|\bar{y})$ (see (6) and recall that $\bar{y}=\{\overline{y_n}\}_{n\in G}$).

4.3 Lemma. Suppose that $\mathfrak{F}\in S_p$ and $Q\in\mathfrak{F}_\otimes$. If $(x,y)\in l_p\times l_{p'}$ then $\langle Qx,y\rangle = \langle x, Q'y\rangle$.

Proof. Suppose that $Q=A_\otimes$ and let l_0 denote the set of all sequences $x=\{x_n\}_{n\in G}$ such that $x_n=0$ whenever $|n|$ is sufficiently large. If a and b are in l_0, then

$$\sum_n\sum_\nu b_n a_\nu [YA]_{n-\nu} = \sum_\nu\sum_n a_\nu b_n [YA']_{\nu-n}. \tag{11}$$

This follows from the relation $[YA]_m = [YA']_{-m}$ (which in turn comes directly from the definition (4)); in view of 4.2, (6), (5) and (11) we therefore have

$$\langle Qa, b\rangle = \langle A \otimes a, b\rangle = \langle a, A' \otimes b\rangle = \langle a, Q'b\rangle \tag{12}$$

when $(a, b) \in l_0 \times l_0$. Since l_0 is dense in any space l_p, there exist sequences $\{a^{(n)}\}_n$ and $\{b^{(k)}\}_k$ with $(a^{(n)}, b^{(k)}) \in l_0 \times l_0$ such that

$$\lim_{n\to\infty} \|x - a^{(n)}\|_p = 0, \qquad \lim_{k\to\infty} \|y - b^{(k)}\|_{p'} = 0. \tag{13}$$

But $Q_p \in \mathfrak{E}_p$ and $(Q')_{p'} \in \mathfrak{E}_{p'}$ (since $Q' = B_\otimes$ and $B = A' \in \mathfrak{F} \in S_p$). From (13) therefore follows that

$$\langle Q_p x, y\rangle = \lim\lim \langle Q_p a^{(n)}, b^{(k)}\rangle,$$

$$\langle x, (Q')_{p'} y\rangle = \lim\lim \langle a^{(n)}, (Q')_{p'} b^{(k)}\rangle \qquad (n\to\infty,\ k\to\infty).$$

Since $Q_p x = Qx$ and $(Q')_{p'} y = Q'y$, a reference to (12) completes the proof.

4.4 Theorem. *Suppose that $\mathfrak{F} \in S_p$ and $Q \in \mathfrak{F}_\otimes$. If $(x, y) \in l_p \times l_{p'}$ then $(Qx|b) = (x|\overline{Q}b)$.*

Proof. Note first that

$$[Q'\bar{b}]_\nu = \sum_n \bar{b}_n [YA']_{\nu-n} = \bar{I}\Big\{\sum_n b_n \bar{I}\big([YA']_{\nu-n}\big)\Big\}$$

where for typographical reasons $\bar{I}\lambda$ denotes $\bar{\lambda}$. But from the definition (4) results easily that $\bar{I}([YA']_m) = [Y\bar{A}]_m$, and it can therefore be concluded from 4.2 and 4.3 that

$$(Qx|b) = \langle Qx, \bar{b}\rangle = \langle x, Q'\bar{b}\rangle = \sum_\nu x_\nu \bar{I}\Big\{\sum_n b_n [Y\bar{A}]_{\nu-n}\Big\} = (x|\overline{Q}b).$$

5. The main results

Denote by $(\mathfrak{F}R)$ the set of all real-valued functions in $\mathfrak{F}$. If $\mathfrak{F} \in S_p$ and $Q \in (\mathfrak{F}R)_\otimes$, then $Q = \overline{Q}$ and from 4.4 follows that

$$(Qx|b) = (x|Qb) \quad \text{for all} \quad (x, b) \quad \text{in} \quad l_p \times l_{p'}. \tag{14}$$

Thus $(\mathfrak{F}R)_\otimes \subset \mathscr{R}(l_p)$ (see § 1). If $Q \in \mathfrak{F}_\otimes$, then it is easily seen that $Q = B_\otimes + iC_\otimes$, where both B and C are in $(\mathfrak{F}R)_\otimes$. Thus $\mathfrak{F}_\otimes \subset \mathscr{N}(l_p)$ (see § 1).

Accordingly, if T is in the set $\mathfrak{F}_{\otimes p} = \{A_{\otimes p} : A \in \mathfrak{F}\}$, then $T = A_{\otimes p}$ for some A in $\mathfrak{F}$, and T has the extension $Q = A_\otimes \in \mathfrak{F}_\otimes$ which belongs to $\mathscr{N}(l_p)$. We thus obtain

5.1 Theorem. *If $\mathfrak{F} \in S_p$, then $\mathfrak{F}_{\otimes p}$ consists of vaguely normal operators.*

5.2 Remarks. If $\mathfrak{F} \in S_2$, then the members of $(\mathfrak{F}R)_{\otimes p}$ are selfadjoint operators, and $\mathfrak{F}_{\otimes 2}$ consists of normal operators. By 4.1, this holds true in the particular case $\mathfrak{F} = L^\infty(X)$.

5.3 Notation. *Let $\mathfrak{B}$ be the set of all functions of finite variation on* $[0, 1]$, *and let $\mathfrak{A}$ be the set of all functions which are equal almost-everywhere to some member of $\mathfrak{B}$.* It has been shown in [*5, 6*] that $\mathfrak{A} \in S_p$; accordingly, all operators in $\mathfrak{A}_{\otimes p}$ are vaguely normal.

If $T \in \mathfrak{E}_p$, the notion of adjoint operator T^* of T is a classical one [*15*, p. 169]. The usual identifications yield

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \qquad \text{when } (x, y) \in l_p \times l_{p'}.$$

If $T \in \mathfrak{A}_{\otimes p}$, then $T = A_{\otimes p}$ for some A in $\mathfrak{A}$, and it follows from 4.3 that

$$T^* = (A')_{\otimes p'}. \tag{15}$$

Let $\Pi(A)$ denote the set of all complex λ such that $\mu(\{\vartheta : A(\vartheta) = \lambda\}) \neq 0$. HARTMAN [*3*] has pointed out that, when $1 < p \leqq 2$, then

$$P\sigma(B_{\otimes p}) \subset \Pi(B) \subset P\sigma(B_{\otimes p'}) \qquad (\text{if } B \in \mathfrak{A}) \tag{16}$$

where $P\sigma(T)$ denotes the point-spectrum of T.

5.4 Theorem. *If* $2 \leqq q < \infty$ *and* $T \in \mathfrak{A}_{\otimes q}$, *then the resolvent set* $Rs(T)$ *is void.*

Proof. By hypothesis there exists a member A of $\mathfrak{A}$ such that $T = A_{\otimes q}$. Observe that

$$Rs(T) \subset P\sigma(A'_{\otimes q'}) \subset \Pi(A') = \Pi(A) \subset P\sigma(A_{\otimes q});$$

the first inclusion follows from (15) and the fact that $Rs(T) \subset P\sigma(T^*)$ (see [*11*, p. 304, Theorem 5]), the second and third inclusion is obtained from (16) by setting $p = q'$, while the equality is easily established*. We have thus obtained $Rs(T) \subset P\sigma(T)$, whence the conclusion $Rs(T) = 0$ (since Rs and $P\sigma$ are always disjoint).

5.5. From 5.2 follows that $A_{\otimes 2}$ is a normal operator (in the usual sense) whenever $A \in \mathfrak{A}$, and all the operators encountered in § 3 are normal operators. Suppose that $T \in \mathfrak{A}_{\otimes p}$; $T = A_{\otimes p}$, $A \in \mathfrak{A}$. Let E^T be the resolution of the identity for $A_{\otimes 2}$. From 3.3 and [*2*, p. 71, Theorem 1] therefore follows that

$$\int \lambda \cdot M^A(d\lambda) = \int \lambda \cdot E^T(d\lambda).$$

Since both M^A and E^T are compact complex spectral measures, it follows that $M^A = E^T$ (see [*2*, p. 65]). If A is now the function $I(\vartheta) = \vartheta$ and $J = I_{\otimes p}$, then the same reasoning (again using 3.3) yields $M = E^J$. A small computation will verify that $-2\pi i J$ is the operator H defined by (1); consequently $E^H = -2\pi i M$.

Now $(b, c) \in l_2 \cap (l_p \times l_{p'})$ implies $(A \otimes b \mid c) = (Tb \mid c)$ and 3.2 now yields the relation

$$(Tb \mid c) = \int \lambda \cdot (E^T(d\lambda)\, b \mid c) = \int_0^1 \frac{iA(\vartheta)}{2\pi} (E^H(d\vartheta)\, b \mid c)$$

for all pairs (b, c) in $l_2 \cap (l_p \times l_{p'})$. The equation (2) in the introduction is intended as a concise rendering of the above.

5.6 Corollary. Suppose that $T \in \mathfrak{A}_{\otimes p}$ and let $\sigma(T)$ denote the spectrum of T. HARTMAN [*3*] has shown that there exists a function B in $\mathfrak{B}$ such that $T = B_{\otimes p}$ and $\sigma(B_{\otimes 2}) = \sigma(T) = \varpi(B)$ (see 2.5). As in 5.5, let E^T be the resolution of the identity for $B_{\otimes 2}$. Accordingly, if g is a bounded Borel function defined on $\sigma(T)$ with $(g \circ B) \in \mathfrak{A}$, then $(g \circ B)_{\otimes p}$ is an operator V which (in view of 3.2 and 5.5)

* Note that $\{\vartheta : A'(\vartheta) = \lambda\} = \{1 - \alpha : A(\alpha) = \lambda\}$ and $\mu(E) \neq 0$ if and only if $\mu(\{1 - \alpha : \alpha \in E\}) \neq 0$.

satisfies the relation
$$(Vb|c) = \int_{\sigma(T)} g(\lambda)\cdot(E^T(d\lambda)\,b|c) \tag{17}$$
for all pairs (b, c) in $l_2 \cap (l_p \times l_{p'})$. In particular, if $1 < p \leqq 2$ and $c = \{\delta_{nk}\}_{k \in G}$ (see 3.4), then
$$[Vb]_n = \int_{\sigma(T)} g(\lambda)\cdot[E^T(d\lambda)\,b]_n \tag{18}$$
holds for all b in l_p and all n in G.

5.7 Examples. Consider the translation operator $x \to \tau x$ defined on l_p, where $\tau x = \{x_{n+1}\}_{n \in G}$. It is easily checked that $\tau = A_{\otimes p}$, where $A(\vartheta) = \exp(-2\pi i \vartheta)$. It was proved in [*6*] that, if $A(\vartheta) = \sum \lambda_m \vartheta^m$ is an entire function, then $A_{\otimes p} = \sum \lambda_m (I_{\otimes p})^m$. Accordingly, we may write
$$\tau = \sum_{m=0}^{\infty} \frac{1}{m!}(-2\pi i I_{\otimes p})^m = \exp H$$
since $H = -2\pi i I_{\otimes p}$. Let $\log\lambda$ be the branch of the logarithm such that $\log\exp(i\lambda) = i\lambda$ for $-2\pi < \lambda \leqq 0$; then $I = (i/2\pi)(\log\circ A)$, whence $I_{\otimes p} = (i/2\pi)(\log\circ A)_{\otimes p}$, and from (18) it follows that the curious relation
$$[Hx]_n = \sum_{\substack{\nu\in G\\ \nu\neq n}} x_\nu \frac{1}{n-\nu} = \oint_{|\lambda|=1} \log\lambda\cdot[E^\tau(d\lambda)\,b]_n$$
holds for all n in G and all x in l_p (when $1 < p \leqq 2$).

If we take $A^{(\alpha)}(\vartheta) = \exp(-2\pi i\alpha\vartheta + \pi i\alpha)$, then (when $|\alpha|$ is not an integer);
$$[A^{(\alpha)} \otimes x]_n = \frac{\sin\alpha\pi}{\pi} \sum_{\nu=-\infty}^{\infty} x_\nu \frac{1}{n-\nu+\alpha} \qquad (n \in G).$$
Both operators H and $(A^{(\alpha)})_{\otimes p}$ have been studied in a famous article by M. Riesz [*11*]. If $\alpha = 1$, then $A^{(\alpha)} = -A$, so that
$$\tau = -(A^{(1)})_{\otimes p}.$$

6. The weak closure theorem

In case $p = 2$ it is customary to denote by $g(T)$ the operator V defined by (17) for all pairs (b, c) in $l_2 \times l_2$; therefore $(g\circ A)_{\otimes 2} = g(A_{\otimes 2})$. In particular, to any T in $\mathfrak{A}_{\otimes 2}$ corresponds a bounded Borel function g such that $T = g(I_{\otimes 2})$ (this is obtained by taking $A = I$ and $T = g_{\otimes 2}$). In the general case ($p \neq 2$), the result $\tau = \exp(-2\pi i I_{\otimes p})$ of 5.7 illustrates the fact that, if $\mathfrak{F}$ is the class of absolutely continuous functions, then $\mathfrak{F}_{\otimes p}$ is in the uniform closure of the algebra generated by $I_{\otimes p}$ (this was proved in [*5*]). A weaker result relating to the whole algebra $\mathfrak{A}_{\otimes p}$ will now be established.

6.1. The algebra $\mathfrak{U}$ generated by $I_{\otimes p}$ is the (complex) linear span of the set $\{(I_{\otimes p})^n : n = 0, 1, 2, \ldots\}$. A member T of $\mathfrak{E}_p$ is in the weak closure of $\mathfrak{U}$ if there exists a sequence $\{T_\iota\}_\iota$ in $\mathfrak{U}$ such that $(Tx|y) = \lim(T_\iota x|y)$, $\iota\to\infty$ when $(x, y) \in l_p \times l_{p'}$.

6.2. Two more remarks will be in order. From the definition (5) follows that $A_{\otimes p} = B_{\otimes p}$ if $A = B$ μ-almost-everywhere; accordingly, $\mathfrak{A}_{\otimes p} = \mathfrak{B}_{\otimes p}$ (see 5.3). It is clear from Stečkin's paper [*13*] that the isomorphism $A \to A_{\otimes p}$ of $\mathfrak{B}$ into $\mathfrak{E}_p$

is continuous in the following sense: there exists a number k_0 such that

$$\|A_{\otimes p}\|_p \leqq k_0 \|A\|_v, \tag{19}$$

where $\|A\|_v$ denotes the total variation of A.

6.3 Theorem. *If $T \in \mathfrak{A}_{\otimes p}$, then T is in the weak closure $\overline{\mathfrak{U}}$ of the algebra generated by $I_{\otimes p}$.*

Proof. If $T \in \mathfrak{A}_{\otimes p}$, then $T \in \mathfrak{B}_{\otimes p}$ (see 6.2) and we can write $T = f_{\otimes p} + i g_{\otimes p}$ with $f_{\otimes p}$ and $g_{\otimes p}$ in $(\mathfrak{B}R)_{\otimes p}$ (see § 5). It is easily seen that any complex linear combination of members of $\overline{\mathfrak{U}}$ belongs to $\overline{\mathfrak{U}}$; since T is a combination of members of $(\mathfrak{B}R)_{\otimes p}$, it follows that, in order to prove $T \in \overline{\mathfrak{U}}$, it will suffice to establish $(\mathfrak{B}R)_{\otimes p} \subset \overline{\mathfrak{U}}$. To that effect, write $T = (P_0)_{\otimes p}$, where P_0 is a real-valued function in $\mathfrak{B}$. Since P_0 is continuous μ-almost-everywhere, we have

$$P_0 = \lim_{\iota \to \infty} P_\iota \qquad (\mu\text{-almost-everywhere}) \tag{20}$$

where P_ι is the ι^{th} Bernstein polynomial of P_0 [*8*, p. 4]. But, from $P_0 \in (\mathfrak{B}R)$ follows [*8*, p. 25 or 9, Satz 6] that

$$\sup\{\|P_\iota\|_v : \iota \neq 0\} = M_0 < \infty. \tag{21}$$

Take any b in the set l_0 of all sequences b such that $b_n = 0$ when $|n|$ is sufficiently large. Now $|P_\iota| \leqq \|P_\iota\|_v \leqq k_0$ for $\iota \neq 0$; take any m in G, then the measure N_m defined in 3.4 enables us to write

$$\int P_0 \cdot dN_m = \lim_{\iota \to \infty} \int P_\iota \cdot dN_m$$

from LEBESGUE's dominated-convergence theorem (recall that $N_m \ll \mu$, so that (20) holds N_m-almost-everywhere). From 3.4 therefore:

$$[P_0 \otimes b]_m = \lim_{\iota \to \infty} [P_\iota \otimes b]_m \qquad (b \in l_0,\ m \in G). \tag{22}$$

The second part of the proof begins by taking an arbitrary x in l_p and observing that

$$\|P_\iota \otimes x\|_p \leqq \|x\|_p \cdot k_0 \cdot M_0 \qquad (\iota \neq 0) \tag{23}$$

(from (19) and (21)). Fix m for the time being and consider the linear functionals $\varphi_\iota = \{x \in l_p \to [P_\iota \otimes x]_m\}$; from (23) follows that $\sup\{\|\varphi_\iota\| : \iota \neq 0\} \leqq k_0 M_0$. Since moreover $\varphi_0(b) = \lim \varphi_\iota(b)$ (from (22)) for all b in the set l_0 which is dense in l_p, it results that $\varphi_0(x) = \lim \varphi_\iota(x)$ $(\iota \to \infty)$ when $x \in l_p$ [*10*, p. 131]. In other words; for any x in l_p, the property $[P_0 \otimes x]_m = \lim [P_\iota \otimes x]_m$ holds for all m in G. In view of (23) and [*10*, p. 139, Satz 6], this in turn implies that

$$(P_0 \otimes x | y) = \lim_{\iota \to \infty} (P_\iota \otimes x | y) \qquad \text{when } (x, y) \in l_p \times l_{p'}.$$

Now P_ι is a polynomial $\sum \lambda_n I^n$ $(\iota \neq 0)$ and therefore $P_\iota \otimes x = T_\iota x$, where $T_\iota = (P_\iota)_{\otimes p} = \sum \lambda_n (I_{\otimes p})^n$ (since the mapping $A \to A_{\otimes p}$ is a homomorphism, as pointed out in 5.3). But then the T_ι are in the linear span $\mathfrak{U}$, and since $Tx = P_0 \otimes x$:

$$(Tx | y) = \lim_{\iota \to \infty} (T_\iota x | y) \qquad \text{when } (x, y) \in l_p \times l_{p'},$$

which by 6.1 concludes the proof.

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References

[1] Halmos, P. R.: Measure Theory. New York 1950.
[2] Halmos, P. R.: Introduction to Hilbert Space and the Theory of Spectral Multiplicity. New York 1951.
[3] Hartman, P.: On Laurent operators on l_p. Proc. American Math. Soc. **8**, No. 1, 45–48 (1957).
[4] Hartman, P. & A. Wintner: On the spectra of Toeplitz's matrices. American J. of Math. **72**, 359–366 (1950).
[5] Krabbe, G. L.: Spectral isomorphisms for some rings of infinite matrices on a Banach space. American J. of Math. **78**, 42–50 (1956).
[6] Krabbe, G. L.: On the spectra of certain Laurent matrices. Proc. American Math. Soc. **8**, No. 5, 894–897 (1957).
[7] Loomis, L. H.: An Introduction to Abstract Harmonic Analysis. New York 1953.
[8] Lorentz, G. G.: Bernstein Polynomials. Toronto 1953.
[9] Lorentz, G. G.: Zur Theorie der Polynome von S. Bernstein. Matematicheskii Sbornik **44**, 543–556 (1937).
[10] Ljusternik, L. A., & W. I. Sobolew: Element der Funktionalanalysis. Berlin 1955.
[11] Riesz, M.: Sur les fonctions conjuguées. Math. Z. **27**, 218–244 (1928), in particular, 241–243.
[12] Riesz, F.: Les systèmes d'équations linéaires à une infinité d'inconnues. Paris 1913.
[13] Stečkin, S. B.: On bilinear forms. Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS **71**, 237–240 (1950).
[14] Titchmarsh, E. C.: Reciprocal formulae involving series and integrals. Math. Z. **25**, 321–347 (1926).
[15] Zaanen, A. C.: Linear Analysis. New York-Amsterdam 1953.

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Sulle Funzioni di più Variabili a Variazione Limitata

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Memoria presentata da G. FICHERA

Il presente lavoro è originato dalle lezioni tenute negli anni accademici 1956—57 e 1957—58 dal Prof. FICHERA presso l'Istituto Nazionale di Alta Matematica. Esso ha per oggetto la dimostrazione della equivalenza di diverse definizioni di funzione di più variabili a variazione limitata.

Prima di enunciare il tipo di problemi considerati in questo lavoro è opportuno introdurre alcune convenzioni che ci permetteranno di rendere più spedito il discorso.

Indicato con $S^{(r)}$ lo spazio euclideo ad r dimensioni, denoteremo con P il punto generico di $S^{(r)}$ e con $x_1, \dots, x_r$ le coordinate di P rispetto ad un fissato sistema cartesiano ortogonale. Sia h un intero tale che $1 \leqq h \leqq r$; indichiamo con $\alpha^{(h)}$ una combinazione di classe h dei numeri $1, 2, \dots, r$ i cui elementi $a_1, \dots, a_h$ pensiamo disposti in ordine crescente: $a_1 < \dots < a_h$. Se non è $h = r$, indicheremo con $b_1, \dots, b_{r-h}$ l'$(r-h)$-upla dei numeri interi compresi tra 1 ed r, non contenuti nella combinazione $\alpha^{(h)}$ e tali che $b_1 < \dots < b_{r-h}$. Indicheremo inoltre con ξ e η rispettivamente una h-upla ed una $(r-h)$-upla di numeri reali e precisamente:

$$(1_{\alpha^{(h)}}) \qquad \xi \equiv (x_{a_1}, \dots, x_{a_h}), \qquad \eta \equiv (x_{b_1}, \dots, x_{b_{r-h}}).$$

Supposto $1 < h < r$, se u è una funzione definita in un insieme di $S^{(r)}$, converremo di indicare $u(P)$ anche con il simbolo $u(\xi, \eta)$.

Sia t una variabile, prescelta fra le $x_{a_1}, \dots, x_{a_h}$ che costituiscono la h-upla ξ. Se è $h > 1$, indicheremo con y l'$(h-1)$-upla costituita dalle $h-1$ rimanenti variabili (sempre disposte in ordine di indice crescente). Nel seguito, dovendo esplicitamente indicare la dipendenza della funzione u dalla variabile t adopereremo, per denotare la $u(P)$, anche il simbolo $u(t, y, \eta)$. Se $h = 1$, si converrà porre: $u(P) = u(t, \eta) \equiv u(\xi, \eta)$; se $h = r$ invece: $u(P) \equiv u(t, y) \equiv u(\xi)$. Inoltre, se I è un intervallo di $S^{(r)}$ [1], indicheremo con I_t la proiezione di I sull'asse $S_t^{(1)}$ della variabile t, cioè l'intervallo descritto da t in $S_t^{(1)}$, quando P varia in I. Analogamente indicheremo con I_ξ la proiezione di I sulla varietà lineare $S_\xi^{(h)}$ di equazioni $x_{b_1} = 0, \dots, x_{b_{r-h}} = 0$, con I_η (se $h < r$) la proiezione di I sulla varietà lineare $S_\eta^{(r-h)}$ di equazioni $x_{a_1} = 0, \dots, x_{a_h} = 0$ ed infine, se $h > 1$, con I_y la proiezione di I sulla varietà lineare $S_y^{(h-1)}$ di equazioni $t = 0$, $x_{b_1} = 0, \dots, x_{b_{r-h}} = 0$. Se

[1] Usando il termine *intervallo* intenderemo sempre riferirci, salvo esplicito avviso in contrario, ad intervalli superiormente aperti; pertanto, con la locuzione *intervallo di* $S^{(r)}$ intendiamo un insieme dello spazio euclideo ad r dimensioni definito da condizioni del tipo $c_i \leqq x_i < d_i$ con $d_i > c_i$ $(i = 1, \dots, r)$.

$1<h<r$, i punti di $S^{(r)}$ verranno indicati, secondo che più torni comodo, con uno dei due simboli, (t, y, η) e (ξ, η), i punti di $S_\xi^{(h)}$ con ξ oppure con (t, y), i punti dell'iperpiano $S_{y,\eta}^{(r-1)}$, d'equazione $t=0$, con (y, η).

Se $h=1$, per i punti di $S^{(r)}$ verranno usati i simboli (t, η) oppure (ξ, η), per i punti di $S_\xi^{(1)}$ i simboli t o ξ. Se $h=r$, per i punti di $S^{(r)}$ adotteremo i simboli ξ oppure (t, y). In tutti i casi, per i punti di $S_t^{(1)}$ useremo la lettera t.

Se $1<h<r$, indicato con I un intervallo di $S^{(r)}$, con $I_y \times I_\eta$ denoteremo l'intervallo di $S_{y,\eta}^{(r-1)}$, descritto da (y, η) al variare di y in I_y e di η in I_η.

Quando, nel seguito, in corrispondenza ad una funzione $u(P) \equiv u(x_1, \ldots, x_r)$, considereremo una coppia di interi h e k con $1 \leq h \leq r$, $1 \leq k \leq r$, converremo sempre di fare le posizioni $(1_{\alpha(h)})$ e di assumere $t=x_k$. Indicheremo inoltre con τ la $(r-1)$-upla $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_r)$.

Ciò posto, una funzione $u(P)$, definita nell'involucro dell'intervallo I di $S^{(r)}$ [2] ed ivi continua, dicesi *a variazione limitata secondo* TONELLI *rispetto ad* x_k, oppure, come sempre, diremo nel seguito, $[CVL]_{\alpha(1),k}$ se:

1°) fissato il punto η quasi ovunque in I_η [3] la funzione di t: $u(t, \eta)$ è VL in I_t;

2°) la funzione $V(\eta)$ che coincide con la variazione totale dell'anzidetta funzione nei punti η di I_η in corrispondenza ai quali $u(t, \eta)$ è VL e che assume il valore zero nei restanti punti di I_η, è sommabile nell'intervallo I_η [4].

Il Prof. FICHERA ha dato la seguente definizione di funzione a variazione limitata [5]:

Una funzione $u(P)$, definita nell'involucro dell'intervallo I di $S^{(r)}$ ed ivi continua, dicesi a variazione limitata rispetto a $x_k=t$ se, fatte le posizioni $(1_{\alpha(r)})$ e posto, in corrispondenza ad ogni intervallo r-dimensionale $J \subset I$: $F(J) = \int_{\mathscr{F}J} u(\xi)\, \nu_t\, d\omega$ (ove $\mathscr{F}J$ indica la frontiera di J, ν_t la componente secondo l'asse t del versore normale esterno ad $\mathscr{F}J$, $d\omega$ è l'elemento di misura ipersuperficiale su $\mathscr{F}J$), la funzione d'intervallo $F(J)$ è VL in I [6].

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[2] Per involucro di un insieme I s'intende l'insieme dei punti che hanno distanza nulla da I. L'involucro di I sarà indicato con $\bar{I}$.

[3] Le locuzioni ,,quasi ovunque" ,,funzione misurabile" ,,funzioni equivalenti" sono qui sempre da intendersi nel senso di Lebesgue.

[4] La funzione $V(\eta)$ è misurabile in I_η; *cfr.* [6] p. 430.

[5] *Cfr.* [6] p. 428.

[6] Sia $F(J)$ una funzione definita nella famiglia $\{J\}$ degli intervalli contenuti in I. La funzione $F(J)$ dicesi a *variazione limitata* in I se, considerata un'arbitraria decomposizione δ di I in intervalli di $S^{(r)}$: $I = J_1 + \cdots + J_m$ e posto $\sigma(\delta) = \sum_{i=1}^{m} |F(J_i)|$, la variabile $\sigma(\delta)$ riesce superiormente limitata al variare comunque di δ. L'estremo superiore di $\sigma(\delta)$ prende il nome di *variazione totale* e sarà indicata con $V_F(I)$. È evidente che, se F è VL in I, essa è tale anche in ogni $J \subset I$. Resta cosi definita in $\{J\}$ la funzione $V_F(J)$, variazione totale di $F(J)$ in J, la quale è non negativa ed additiva se F è additiva. Una funzione $F(J)$, additiva e VL, è derivabile in quasi tutti i punti di I e la sua derivata F' è misurabile e sommabile in I. Nel seguito, usando il simbolo F' o parlando di derivata di una funzione d'intervallo F, intenderemo sempre riferirci alla derivata calcolata nel modo seguente: se P è un punto di I, indicato con $I_{p,l}$, l'intersezione di I con l'intervallo quadrato avente come estremo inferiore P ed il lato di lunghezza l, porremo: $F'(P) = \lim_{n\to\infty} n^r F(I_{p,\frac{1}{n}})$ se questo limite esiste finito e $F'(P)=0$ nell'altro caso. Analoga convenzione faremo per le derivate delle funzioni di una variabile.

Diremo che una funzione u, verificante tale condizione, è $[CVL]_{\alpha^{(r)},k}$ in I.

Il Prof. FICHERA ha dimostrato, nel caso $r=2$, l'equivalenza fra le due definizioni di funzione a variazione limitata ora ricordate[7]. Questa dimostrazione è stata estesa al caso di r qualsiasi in un mio recente lavoro[8]. Il Prof. FICHERA, nei predetti corsi di Analisi tenuti all'Istituto Nazionale di Alta Matematica, ha generalizzato le definizioni $[CVL]_{\alpha^{(1)},k}$, $[CVL]_{\alpha^{(r)},k}$ considerando funzioni soltanto sommabili anzichè funzioni continue e precisamente ha dato le seguenti definizioni:

A. Una funzione $u(P)$, definita nell'involucro dell'intervallo I di $S^{(r)}$, la quale sia ivi misurabile e sommabile, dicesi $[VL]_{\alpha^{(1)},k}$ se, tra le funzioni ad essa equivalenti[9] ve n'è almeno una — che, per semplicità di notazione, seguiteremo ad indicare con u — tale che risultino soddisfatte le seguenti condizioni:

a) per ogni $\eta \in I_\eta$ la funzione di t: $u(t,\eta)$ è VL in I_t;

b) la funzione $V(\eta)$, che rappresenta la variazione totale dell'anzidetta funzione, è maggiorata da una funzione misurabile e sommabile in I_η[10].

Si dimostra (*cfr.* teor. XII) che le derivate parziali rispetto a t delle varie funzioni equivalenti ad u e soddisfacenti le a) e b) sono misurabili ed equivalenti tra loro in I. Allora si assume, per definizione, come *derivata di u rispetto a t* la derivata parziale rispetto a t di una qualsiasi delle funzioni equivalenti ad u e verificanti le a) e b). Questa derivata sarà indicata con il simbolo: $\left[\frac{\partial u}{\partial x_k}\right]_{\alpha^{(1)},k}$.

B. Una funzione $u(P)$, definita nell'involucro dell'intervallo I di $S^{(r)}$, la quale sia ivi misurabile e sommabile, è $[VL]_{\alpha^{(r)},k}$ se, tra le funzioni ad essa equivalenti, ve n'è almeno una — che per semplicità seguiteremo ad indicare con u — tale che, posto per ogni intervallo r-dimensionale $J \subset I$: $\Phi_{\alpha^{(r)}}(J) = \int_{\mathcal{F}I} u(\xi)\, \nu_t\, d\omega$[11], la funzione $\Phi_{\alpha^{(r)}}(J)$ risulta VL in I.

Si dimostra (*cfr.* teor. XIII) che, se u ed $\bar{u}$ sono due funzioni equivalenti e verificanti la detta condizione, le corrispondenti derivate $\Phi'_{\alpha^{(r)}}$ e $\bar{\Phi}'_{\alpha^{(r)}}$ sono equivalenti in I. Allora si assume, per definizione come *derivata di u rispetto a t*, la derivata $\Phi'_{\alpha^{(r)}}$ relativa ad una qualsiasi delle funzioni equivalenti ad u e verificanti la condizione in discorso. Tale derivata verrà indicata con il simbolo $\left[\frac{\partial u}{\partial t}\right]_{\alpha^{(r)},k}$[12].

Il Prof. FICHERA ha poi anche proposto la seguente generale definizione di funzione a variazione limitata[13], che include quella di funzione $[VL]_{\alpha^{(1)},k}$ e quella di funzione $[VL]_{\alpha^{(r)},k}$:

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[7] *Cfr.* [*6*] p. 432.

[8] *Cfr.* [*5*] p. 100.

[9] Due funzioni si diranno equivalenti se sono quasi ovunque eguali.

[10] Questa definizione è equivalente a quella di funzione a variazione limitata generalizzata secondo CESARI-TONELLI; *cfr.* [*4*] p. 299, [*18*] p. 316.

[11] Si tenga presente che, per il noto teorema di Fubini, tra le funzioni equivalenti ad una funzione misurabile e sommabile in $\bar{I}$ ve n'è almeno una che risulta misurabile e sommabile su tutte le sezioni coordinate di $\bar{I}$ cioè sulle intersezioni di $\bar{I}$ con iperpiani paralleli ad iperpiani coordinati.

[12] Si dimostra facilmente che ogni funzione u la quale sia $[CVL]_{\alpha^{(1)},k}$ (che sia $[CVL]_{\alpha^{(r)},k}$) è anche $[VL]_{\alpha^{(1)},k}$ (è anche $[VL]_{\alpha^{(r)},k}$) e, viceversa, ogni funzione continua che sia $[VL]_{\alpha^{(1)},k}$ (che sia $[VL]_{\alpha^{(1)},k}$) è anche $[CVL]_{\alpha^{(1)},k}$ (è anche $[CVL]_{\alpha^{(r)},k}$); *cfr.* [*7*].

[13] *Cfr.* [*7*].

fissati gli interi h e k, con $1 \leq h \leq r$, $1 \leq k \leq r$ e la combinazione $\alpha^{(h)}$ contenente l'intero k, una funzione $u(P)$, definita nell'involucro dell'intervallo I di $S^{(r)}$, la quale sia ivi misurabile e sommabile, dicesi $[VL]_{\alpha^{(h)},k}$, se tra le funzioni ad essa equivalenti ve n'è almeno una — che, per semplicità di notazione, seguiteremo ad indicare con u — tale che siano soddisfatte le condizioni:

$1_{\alpha^{(h)}}$) posto, per ogni $\eta \in I_\eta$ e per ogni intervallo J_ξ contenuto in I_ξ:

$$\Phi_{\alpha^{(h)}}(J_\xi, \eta) = \int_{\mathscr{F} J_\xi} u(\xi, \eta)\, \nu_t \, d\omega \text{ [14]},$$

ove $\mathscr{F} J_\xi$ è la frontiera di J_ξ su $S_\xi^{(h)}$, la $\Phi_{\alpha^{(h)}}(J_\xi, \eta)$, come funzione dell'intervallo J_ξ, per ogni fissato η in I_η è VL in I_ξ;

$2_{\alpha^{(h)}}$) la variazione totale $V_{\Phi_{\alpha^{(h)}}}(I_\xi, \eta)$ è maggiorata da una funzione $g_{\alpha^{(h)}}(\eta)$ misurabile e sommabile in I_η.

Le definizioni **A** e **B**, prima ricordate, rientrano in questa generale, ora data, come casi particolari relativi ai valori $h = 1$ ed $h = r$ rispettivamente; si deve solo tener presente che, nel caso di $h = 1$, indicati con α e β gli estremi del generico intervallo J_ξ, si ha $\Phi_{\alpha^{(1)}}(J_\xi, \eta) = u(\beta, \eta) - u(\alpha, \eta)$ [15], mentre $V_{\Phi_{\alpha^{(1)}}}(I_\xi, \eta)$ rappresenta la variazione totale di u in I_ξ. Inoltre, per $h = r$, la condizione $2_{\alpha^{(h)}}$) non è da imporre perchè perde senso[16].

Si dimostrerà (*cfr.* teor. XI) che, se u ed $\bar{u}$ sono due funzioni equivalenti e verificanti la condizione $1_{\alpha^{(h)}}$), le corrispondenti derivate $\Phi'_{\alpha^{(h)}}$, $\bar{\Phi}'_{\alpha^{(h)}}$ sono equivalenti in I. Allora si assumerà per definizione, come derivata di u rispetto a $t \equiv x_k$ la derivata $\Phi'_{\alpha^{(h)}}$, relativa ad una qualsiasi delle funzioni equivalenti ad u e verificanti la $1_{\alpha^{(h)}}$) e $2_{\alpha^{(h)}}$); tale derivata verrà indicata con il simbolo $\left[\dfrac{\partial u}{\partial t}\right]_{\alpha^{(h)},k}$.

L'equivalenza tra le diverse definizioni che si ottengono, facendo variare h da 1 ad r e, per ogni h, la combinazione $\alpha^{(h)}$ in tutti i modi possibili, può dimostrarsi, nel caso delle funzioni continue, estendendo la dimostrazione dell'equivalenza tra $[CVL]_{\alpha^{(1)},k}$ e $[CVL]_{\alpha^{(r)},k}$ cui prima si è accennato[17]. Il Prof. FICHERA ha posto il problema di dimostrare tale equivalenza nel caso generale (cioè senza fare l'ulteriore ipotesi della continuità) e di far vedere che tutte le derivate $\left[\dfrac{\partial u}{\partial t}\right]_{\alpha^{(h)},k}$ che si ottengono, per un fissato k, in corrispondenza alle diverse scelte dell'intero h e della combinazione $\alpha^{(h)}$, differiscono tra loro solo nei punti di un insieme di misura nulla.

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[14] Si tenga presente che, in virtù del ben noto teorema di FUBINI, tra le funzioni equivalenti ad una funzione misurabile e sommabile in $\bar{I}$ ve n'è almeno una la quale è misurabile e $(h-1)$-dimensionalmente sommabile su ogni intersezione di $\bar{I}$ con le varietà $(h-1)$-dimensionali di equazione: $x_{b_1} = \text{cost.}, \ldots, x_{b_{r-h}} = \text{cost.}\ t = \text{cost.}$

[15] Si osservi che $\Phi_{\alpha^{(h)}}(J_\xi, \eta)$ non è altro che il primo membro della formula di GREEN: $\displaystyle\int_{\mathscr{F} J_\xi} u(\xi, \eta)\, \nu_t \, d\omega = \int_{J_\xi} \frac{\partial u}{\partial t}\, d\xi$. Poichè, per $r = 1$, la formula di GREEN diventa: $\displaystyle u(\beta, \eta) - u(\alpha, \eta) = \int_{J_\xi} \frac{\partial u}{\partial t}\, dt$, appare naturale assumere per $\Phi_{\alpha^{(1)}}(J_\xi, \eta)$ l'incremento della u relativo all'intervallo unidimensionale J_ξ.

[16] Si tenga presente che, per $h = r$, la $\Phi_{\alpha^{(h)}}(J_\xi, \eta)$ non dipende più da η; in tal caso pertanto useremo il simbolo $\Phi_{\alpha^{(h)}}(J_\xi)$.

[17] *Cfr.* [6] e [7].

Nelle righe che seguono si daranno alcuni teoremi che risolvono la questione posta.

Poichè le funzioni $[VL]_{\alpha^{(r)},k}$ sono tutte e sole le funzioni dotate di derivata debole rispetto ad x_k[18], i risultati che verranno ora dimostrati permetteranno anche di stabilire l'equivalenza tra la definizione di funzione dotata di derivata debole rispetto ad x_k e quella di funzione $[VL]_{\alpha^{(h)},k}$ $(h=1,\ldots,r-1)$. Inoltre si avrà che la derivata debole locale di u[19] coincide quasi ovunque con $\left[\frac{\partial u}{\partial x_k}\right]_{\alpha^{(h)},k}$. In particolare quindi sarà dimostrata l'equivalenza tra la definizione di funzione dotata di derivata debole rispetto ad x_k e quella di funzione a variazione limitata generalizzata secondo CESARI-TONELLI (*cfr.* nota[10]). Quest'ultimo risultato è stato di recente dimostrato direttamente da K. KRICKEBERG[20]. Dobbiamo anche osservare che l'equivalenza delle definizioni $[VL]_{\alpha^{(1)},k}$ e $[VL]_{\alpha^{(r)},k}$ poteva già considerarsi acquisita combinando il risultato di FICHERA citato in[18] con quella ora ricordato di KRICKEBERG.

I. *Sia $u(P)\equiv u(t,\eta)$ una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$, la quale sia ivi misurabile, sommabile, e verifichi le $1_{\alpha^{(1)}}$), $2_{\alpha^{(1)}}$). Supponiamo inoltre che per ogni $\eta\in I_\eta$, la funzione di t: $u(t,\eta)$ sia continua nei due punti estremi di I_t e sia continua a sinistra in ogni punto interno ad I_t. Allora la u gode delle seguenti proprietà:*

a) per ogni $t\in\bar I_t$ la $u(t,\eta)$ è funzione di η misurabile e sommabile in I_η,

b) indicata con $V(\eta)$ la variazione totale di $u(t,\eta)$ in I_t e posto $\alpha_s^{(n)}=a+s\frac{b-a}{n}$ (ove a e b sono gli estremi di I_t) riesce:

$$V(\eta)=\lim_{n\to\infty}\sum_{s=1}^{n}\left|u(\alpha_s^{(n)},\eta)-u(\alpha_{s-1}^{(n)},\eta)\right|.$$

Come conseguenza di tali proprietà si ha che $V(\eta)$ è misurabile e sommabile in I_η.

Dato che $u(t,\eta)$ è misurabile e sommabile in I, esiste un insieme N, contenuto in $\bar I_t$, di misura lineare nulla, tale che, per $t\in\bar I_t-N$, la funzione di η: $u(t,\eta)$ è misurabile e sommabile in I_η. Sia t un punto di $N-a$; indichiamo con $\{t_m\}$ una successione di punti contenuti in I_t-N tali che riesca $t_m<t$ e $\lim_{m\to\infty}t_m=t$. Allora si ha: $u(t,\eta)=\lim_{m\to\infty}u(t_m,\eta)$ e pertanto $u(t,\eta)$ risulta misurabile in I_η per ogni $t\in\bar I_t-a$, perchè limite di una successione di funzioni ivi misurabili. Inoltre, se $t'\in I_t-N$, riesce: $|u(t_m,\eta)-u(t',\eta)|\leqq g_{\alpha^{(1)}}(\eta)$.

[18] *Cfr.* [7] ove è dimostrato quanto segue: siano A un campo di $S^{(r)}$, u una funzione localmente sommabile in A, V la classe delle funzioni lipschitziane ed aventi supporto contenuto in A; se la u è dotata di derivata debole, cioè se per ogni $v\in V$ si ha: $\int_A u\frac{\partial v}{\partial x_k}dx=-\int_A v\,d\alpha$, la funzione u è $[VL]_{\alpha^{(r)},k}$ in ogni intervallo contenuto con il suo involucro in A e riesce $\alpha(C)=\int_C d\Phi_{\alpha^{(r)}}$ per ogni insieme chiuso e limitato C contenuto in A $\left(\text{ove con il simbolo }\int_C d\Phi_{\alpha^{(r)}}\right.$ si è inteso l'integrale di RIEMANN-STIELTJES rispetto a $\left.\Phi_{\alpha^{(r)}}\right)$. Se, viceversa, la u è $[VL]_{\alpha^{(r)},k}$, esiste una $\bar u$ equivalente ad u la cui corrispondente $\Phi_{\alpha^{(r)}}$ è una misura la quale verifica la relazione $\int_A u\frac{\partial v}{\partial x_k}=$ $=-\int_A v\,d\Phi_{\alpha^{(r)}}$ per ogni $v\in V$. *Cfr.* anche [*13*].

[19] Per derivata debole locale di u si intende la derivata della misura α (*cfr.* nota (17)).

[20] *Cfr.* [*10*].

Ne viene: $|u(t_m, \eta)| \leqq |u(t', \eta)| + g_{\alpha^{(1)}}(\eta)$ e quindi la funzione $u(t, \eta) = \lim_{m\to\infty} u(t_m, \eta)$ risulta sommabile in I_η. In modo analogo si dimostra che $u(a, \eta)$ è funzione di η misurabile e sommabile in I_η.

È così provata la proprietà a).

Indichiamo con δ una arbitraria decomposizione di I_t in intervalli[21] e con δ_n la decomposizione di I_t, ottenuta mediante i punti $\alpha_s^{(n)}$ $(s = 1, \ldots, n-1)$. Se $\alpha_0 \equiv a, \alpha_1, \ldots, \alpha_m \equiv b$, con $\alpha_0 < \cdots < \alpha_m$, sono i punti mediante i quali è ottenuta la decomposizione δ, porremo $\sigma_\eta(\delta) = \sum_{s=1}^{m} |u(\alpha_s, \eta) - u(\alpha_{s-1}, \eta)|$. Per dimostrare la proprietà b) ci basta far vedere che, assegnati ad arbitrio il numero positivo ε e la decomposizione δ, esiste un intero $n_\varepsilon(\eta)$ tale che $\sigma_\eta(\delta) \leqq \sigma_\eta(\delta_{n_\varepsilon(\eta)}) + \varepsilon$.

Sia ϱ la minima ampiezza degli intervalli relativi alla decomposizione δ. Assumiamo $n_\varepsilon(\eta)$ in modo tale che riesca $\frac{3(b-a)}{n_\varepsilon(\eta)} < \varrho$ ed inoltre, per ogni t verificante la condizione $\alpha_s - \frac{3(b-a)}{n_\varepsilon(\eta)} \leqq t \leqq \alpha_s$, si abbia $|u(\alpha_s, \eta) - u(t, \eta)| < \frac{\varepsilon}{6m}$. È allora facile vedere che, con tali assunzioni, si ha: $\sigma_\eta(\delta) \leqq \sigma_\eta(\delta_{n_\varepsilon(\eta)}) + \varepsilon$.

La misurabilità e sommabilità di $V(\eta)$ è immediata conseguenza delle proprietà a) e b) e del fatto che $\sigma_\eta(\delta_n)$ è non superiore alla funzione sommabile $g_{\alpha^{(1)}}(\eta)$.

II. *Sia $u(P)$ una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$, la quale sia ivi misurabile e verifichi la proprietà $1_{\alpha^{(1)}}$). Allora esiste una funzione $\tilde{u}(t, \eta)$, definita nell'involucro di I, equivalente ad u, la quale, per ogni $\eta \in I_{\eta}$, è funzione di t continua nei due punti estremi di I_t e continua a sinistra in ogni punto interno ad I_t. Inoltre, indicata con $V(\eta)$ (con $\tilde{V}(\eta)$) la variazione totale di $u(t, \eta)$ (di $\tilde{u}(t, \eta)$) in I_t, si ha $\tilde{V}(\eta) \leqq V(\eta)$.*

Indichiamo con a, b gli estremi di I_t, con J_t un intervallo contenuto in I_t avente per estremi i punti a, t, con $v(t, \eta)$ la variazione totale di $u(t, \eta)$, rispetto a t, nell'intervallo J_t e poniamo $v(a, \eta) = 0$.

Definiamo le due funzioni $p(t, \eta)$ e $q(t, \eta)$ al modo seguente:

$$p(t, \eta) = \tfrac{1}{2}\left[v(t, \eta) + u(t, \eta) - u(a, \eta)\right]$$

$$q(t, \eta) = \tfrac{1}{2}\left[v(t, \eta) - u(t, \eta) + u(a, \eta)\right].$$

Le funzioni $p(t, n)$ e $q(t, \eta)$ rappresentano rispettivamente la variazione positiva e la variazione negativa di $u(t, \eta)$ rispetto a t nell'intervallo J_t. Per ogni $\eta \in I_\eta$, $t \in \bar{I}_t$, si ha: $u(t, \eta) = u(a, \eta) + p(t, \eta) - q(t, \eta)$, $v(t, \eta) = p(t, \eta) + q(t, \eta)$. Ne viene $V(\eta) = v(b, \eta) = p(b, \eta) + q(b, \eta)$. Le funzioni $p(t, \eta)$ e $q(t, \eta)$, per ogni $\eta \in I_\eta$, sono non negative e monotone non decrescenti rispetto a t, come subito si verifica. Ognuna di esse, allora, presenta solo discontinuità di prima specie rispetto a t. Per ogni fissato η, l'insieme dei punti di discontinuità di $p(t, \eta)$ è finito o, al più, numerabile.

Consideriamo la funzione:

$$\tilde{p}(t, \eta) \begin{cases} = \lim_{t'\to t^-} p(t', \eta) & a < t \leqq b \\ = \lim_{t'\to t^+} p(t', \eta) & t = a. \end{cases}$$

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[21] Usando la locuzione „decomposizione di un intervallo in intervalli" intenderemo sempre riferirci ad una decomposizione di un intervallo nella somma di un numero finito d'intervalli a due a due privi di punti in comune.

È immediato constatare che $\tilde{p}(t, \eta)$, per $\eta \in I_\eta$, è funzione di t continua nei punti $t = a$ e $t = b$, è continua a sinistra in ogni punto t interno ad I_t ed è monotona non decrescente in $\bar{I}_t$. Riesce inoltre: $\tilde{p}(b, \eta) - \tilde{p}(a, \eta) \leqq p(b, \eta)$. Indichiamo con $\tilde{q}(t, \eta)$ la funzione dedotta da $q(t, \eta)$ con lo stesso procedimento con cui si è dedotta $\tilde{p}$ da p. Allora, la funzione $\tilde{u}(t, \eta) = \tilde{p}(t, \eta) - \tilde{q}(t, \eta)$, per ogni $\eta \in I_\eta$, è continua rispetto a t nei punti $t = a$ e $t = b$ ed è continua a sinistra in ogni punto interno ad I_t. Inoltre, la $\tilde{u}(t, \eta)$, per ogni $\eta \in I_\eta$, è funzione di t a variazione limitata in I_t ed indicata con $\tilde{V}(\eta)$ la sua variazione totale in I_t, si ha:

$$\tilde{V}(\eta) \leqq \tilde{p}(b, \eta) - \tilde{p}(a, \eta) + \tilde{q}(b, \eta) - \tilde{q}(a, \eta) \leqq p(b, \eta) + q(b, \eta) = V(\eta).$$

Infine, avendosi: $\tilde{u}(t, \eta) = \lim_{n \to \infty} u\left(t - \frac{1}{n}, \eta\right)$, per $\eta \in I_\eta$, $a < t \leqq b$, ed essendo $u(t, \eta)$ una funzione misurabile in I, si ha che tale risulta anche la $\tilde{u}(t, \eta)$. D'altra parte, per ogni $\eta \in I_\eta$, le funzioni $u(t, \eta)$ e $\tilde{u}(t, \eta)$ sono quasi ovunque eguali in I_t e pertanto $u(t, \eta)$ e $\tilde{u}(t, \eta)$ risultano equivalenti in I.

In tal modo il teorema è completamente dimostrato.

III. *Siano I un intervallo di $S^{(r)}$, $\{J\}$ la famiglia degli intervalli di $S^{(r)}$ contenuti in I, $F(J)$ una funzione additiva e VL in $\{J\}$. Esiste allora una ed una sola misura $\mu(J)$, definita in $\{J\}$, la quale gode della seguente proprietà: in corrispondenza a μ si può determinare un insieme numerabile N_i contenuto nell'asse x_i $(i = 1, \ldots, r)$ in modo tale che, per ogni intervallo $J \in \{J\}$ definito dalle condizioni: $\alpha_i \leq x_i < \beta_i$ $\alpha_i \notin N_i$, $\beta_i \notin N_i$ $(i = 1, \ldots, r)$, riesca $\mu(J) = F(J)$.*

Pensiamo la F definita su tutti gli intervalli di $S^{(r)}$, assumendola nulla in quegli intervalli che non appartangono a $\{J\}$. Sia μ la misura generata dalla funzione F[22]. Indichiamo con M_i $(i = 1, \ldots, r)$ l'insieme numerabile contenuto nell'asse x_i e costituito dai numeri c_i tali che l'iperpiano $x_i = c_i$ sia di discontinuità per la misura μ. Denotiamo inoltre con M_i' $(i = 1, \ldots, r)$ l'insieme numerabile costituito dai punti d_i dell'asse x_i, in corrispondenza ad ognuno dei quali è verificata la seguente circostanza: se $(a_1, \ldots, a_r)$ e $(b_1, \ldots, b_r)$ sono i punti estremi di I, $\{J'\}_i$ è la famiglia degli intervalli così definiti: $a_m \leqq x_m < b_m$ $(m = 1, \ldots, i-1, i+1, \ldots, r)$, $x_i \in (a_i, t_i)$ allora, per ogni i $(i = 1, \ldots, r)$, le funzioni di t_i: $p_F(J')$ e $q_F(J')$ (con $J' \in \{J'\}_i$) sono discontinue nel punto $t_i = d_i$.

Assumiamo $N_i = M_i + M_i'$. Sia J un intervallo contenuto in $\{J\}$, di punti estremi $(\alpha_1, \ldots, \alpha_r)$, $(\beta_1, \ldots, \beta_r)$, tale che $\alpha_i \notin N_i$, $\beta_i \notin N_i$. Se J_n è l'intervallo definito da: $\alpha_i - \frac{1}{n} \leqq x_i < \beta_i + \frac{1}{n}$ $(i = 1, \ldots, r)$, riesce, com'è facile constatare; $\lim_{n \to \infty} F(J_n) = F(J)$. Si ha inoltre: $\mu(\bar{J}) = \mu(J)$. Poichè, d'altra parte, riesce: $\lim_{n \to \infty} F(J_n) = \mu(\bar{J})$, si ha: $F(J) = \mu(J)$.

IV. *Siano $u(P)$ una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$ ed ivi $[VL]_{\alpha^{(r)}, k}$, h un intero positivo minore di r ed $\alpha^{(h)}$ una combinazione contenente k. Indichiamo con $\{J_\eta\}$ la famiglia degli intervalli di $S_\eta^{(r-h)}$ contenuti in I_η e con J l'intervallo di $S^{(r)}$ definito dalle condizioni: $t \in I_t$, $y \in I_y$, $\eta \in J_\eta$. La funzione $\mathscr{V}(J_\eta) = V_{\Phi_{\alpha^{(r)}}}(J)$ è assolutamente continua in $\{J_\eta\}$.*

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[22] Dicendo che μ è la misura generata da $F(I)$ intendiamo che, per ogni insieme C chiuso e limitato di $S^{(r)}$, riesce: $\mu(C) = \int_C dF$, l'integrale a secondo membro essendo considerato nel senso di RIEMANN-STIELTJES.

Supponiamo che la funzione u stessa verifichi la condizione $1_{\alpha(r)}$). Indichiamo con δ una decomposizione di I_t mediante i punti $\alpha_0, \alpha_1, \ldots, \alpha_n$, con $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. La famiglia di tali decomposizioni sarà indicata con $\{\delta\}$.

Fissata una decomposizione $\delta \in \{\delta\}$, per ogni $J_\eta \in \{J_\eta\}$ poniamo:

$$F_s(J_\eta) = \int_{J_\eta} d\eta \int_{I_y} [u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)]\, dy,$$

e inoltre:

$$V_{F_s}(J_\eta) = \int_{J_\eta} \left[\int_{I_y} |u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)|\, dy \right] d\eta.$$

Indichiamo con I_{s, J_η} l'intervallo di $S^{(r)}$ definito dalle condizioni: $\alpha_{s-1} \leqq t < \alpha_s$, $y \in I_y$, $\eta \in J_\eta$. Si ha:

$$(2_s) \qquad V_{F_s}(J_\eta) \leqq V_{\Phi_{\alpha(r)}}(I_{s, J_\eta}).$$

Infatti riesce: $\Phi_{\alpha(r)}(I_{s, J_\eta}) = F_s(J_\eta)$; allora, indicata con Δ una decomposizione di I_{s, J_η}, in intervalli di $S^{(r)}$: $I_{s, J_\eta} = I^1 + \cdots + I^m$ e posto $\sigma(\Delta) = \sum_{i=1}^{m} |\Phi_{\alpha(r)}(I^i)|$ si ha che: $V_{\Phi_{\alpha(r)}}(I_{s, J_\eta})$ è l'estremo superiore di $\sigma(\Delta)$ al variare comunque di Δ, mentre $V_{F_s}(J_\eta)$ coincide con l'estremo superiore di $\sigma(\Delta)$ al variare di Δ soltanto nell'insieme delle decomposizioni di I_{s, J_η}, del tipo $I_{s, J_\eta} = I^1 + \cdots + I^m$ con $I_t^i \equiv (\alpha_{s-1}, \alpha_s)$.

Ciò prova l'asserto.

Sommando membro a membro le (2_s) per $s = 1, \ldots, n$, si ottiene

$$\sum_{s=1}^{n} V_{F_s}(J_\eta) \leqq \sum_{s=1}^{n} V_{\Phi_{\alpha(r)}}(I_{s, J_\eta}) = V_{\Phi_{\alpha(r)}}(J) = \mathscr{V}(J_\eta)$$

e quindi:

$$(3) \qquad \sum_{s=1}^{n} \int_{J_\eta} \left[\int_{I_y} |u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)|\, dy \right] d\eta \leqq \mathscr{V}(J_\eta).$$

Il primo membro di tale diseguaglianza, per ogni fissato $J_\eta \in \{J_\eta\}$, è funzione di δ. Sia $\Psi(J_\eta)$ il suo estremo superiore su $\{\delta\}$. Facciamo vedere che riesce:

$$(4) \qquad \Psi(J_\eta) = \mathscr{V}(J_\eta).$$

Dalla (3) si ha intanto: $\Psi(J_\eta) \leqq \mathscr{V}(J_\eta)$. Inoltre, fissata una decomposizione Δ di J in intervalli di $S^{(r)}$: $J = I^1 + \cdots + I^m$ e posto: $\sigma(\Delta) = \sum_{i-1}^{m} |\Phi_{\alpha(r)}(I^i)|$, si può sempre determinare una decomposizione δ di $\{\delta\}$ tale che

$$\sigma(\Delta) \leqq \sum_{s=1}^{n} \int_{J_\eta} \left[\int_{I_y} |u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)|\, dy \right] d\eta;$$

basta per questo considerare la totalità degli iperpiani d'equazione $t = \alpha_s$ $(s = 0, \ldots, n)$ ognuno dei quali contiene almeno un punto di $\mathscr{J} I^i$ in corrispondenza a qualche i $(i = 1, \ldots, m)$, ed assumere come δ la decomposizione di I_t relativa ai punti $\alpha_0, \ldots, \alpha_m$. In tal modo è provata la (4).

Sia η_0 un punto di I_η ed indichiamo con $I_{\eta_0, l}$ l'intervallo quadrato contenuto in I_η che ha η_0 come estremo inferiore ed il lato di lunghezza l. Dalla (3) si trae:

$$(5) \qquad \sum_{s=1}^{n} \frac{1}{l^{r-h}} \int_{I_{\eta_0, l}} \left[\int_{I_y} |u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)|\, dy \right] d\eta \leqq \frac{1}{l^{r-h}} \mathscr{V}(I_{\eta_0, l}) .$$

Indichiamo con N_δ l'insieme contenuto in I_η e di misura $(r-h)$-dimensionale nulla, tale che, per $\eta_0 \in I_\eta - N_\delta$, riesce

$$\lim_{l \to 0} \frac{1}{l^{r-h}} \int_{I_{\eta_0, l}} \left[\int_{I_y} |u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)|\, dy \right| d\eta$$
$$= \int_{I_y} |u(\alpha_s, y, \eta_0) - u(\alpha_{s-1}, y, \eta_0)|\, dy .$$

Supponiamo anche che, per $\eta \in I_\eta - N_\delta$, esista la derivata $\mathscr{V}'(\eta)$ di $\mathscr{V}(J_\eta)$. Passando al limite nella (5), per $l \to 0$, in corrispondenza ad ogni punto $\eta \in I_\eta - N_\delta$, si ha:

$$\sum_{s=1}^{n} \int_{I_y} |u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)|\, dy \leqq \mathscr{V}'(\eta) ,$$

donde, per ogni $J_\eta \in \{J_\eta\}$:

$$\sum_{s=1}^{n} \int_{J_\eta} \left[\int_{I_y} |u(\alpha_s, y, \eta) - u(\alpha_{s-1}, y, \eta)|\, dy \right] d\eta \leqq \int_{J_\eta} \mathscr{V}'(\eta)\, d\eta .$$

Da questa diseguaglianza si trae: $\Psi(J_\eta) \leqq \int_{J_\eta} \mathscr{V}'(\eta)\, d\eta$. Poichè $\Psi(J_\eta) = \mathscr{V}(J_\eta)$, si ha l'assoluta continuità di $\mathscr{V}(J_\eta)$.

V. *Sia $u(P) \equiv u(t, y)$ una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$ e $[VL]_{\alpha^{(r)}, k}$ in I. Esiste allora una ed una sola misura $\alpha(J)$, definita nella famiglia $\{J\}$ degli intervalli di $S^{(r)}$ contenuti in I, in corrispondenza alla quale è possibile determinare una funzione $\bar{u}$, equivalente ad u in I, tale che:*

$$\bar{\Phi}_{\alpha^{(r)}}(J) \equiv \int_{\mathscr{F} J} \bar{u}\, v_t\, d\omega = \alpha(J) .$$

Supponiamo che la u verifichi la $1_{\alpha^{(r)}}$). Indichiamo con $\alpha(J)$ la misura generata in $\{J\}$ dalla funzione $\Phi_{\alpha^{(r)}}(J) = \int_{\mathscr{F} J} u\, v_t\, d\omega$. Per il teor. III, esiste, per ogni i, un insieme numerabile N_i, contenuto nell'asse x_i, tale che per ogni intervallo $J \in \{J\}$ definito dalle condizioni $a_i \leqq x_i < b_i$, con $a_i \in I_{x_i} - N_i$, $b_i \in I_{x_i} - N_i$, riesce: $\Phi_{\alpha^{(r)}}(J) = \alpha(J)$. È immediato constatare che, in questo caso, per ogni $i \neq k$, l'insieme N_i è vuoto. Siano a un numero contenuto in $I_t - N_k$, R un intervallo contenuto in I_t, avente un estremo nel punto a e l'altro in un punto b, $\{H\}$ la famiglia degli intervalli di $S_y^{(r-1)}$ contenuti in I_y. La funzione di H: $\alpha(R \times H)$ è assolutamente continua in $\{H\}$. Infatti, dato che, per il teorema precedente, la funzione $V_{\Phi_{\alpha^{(r)}}}(R \times H)$ è assolutamente continua in $\{H\}$, indicata con $g(y)$ la derivata della funzione di H: $V_{\Phi_{\alpha^{(r)}}}(I_t \times H)$ e con $\{b_n\}$ una successione di numeri tali che: $\lim_{n \to \infty} b_n = b$, $b_n \in I_t - N_k$, $b_n < b$, si ha, per ogni $H \in \{H\}$:

$$\left| \int_H [u(b_n, y) - u(a, y)]\, dy \right| \leqq \int_H g(y)\, dy .$$

Poichè:

$$\lim_{n\to\infty}\left|\int_H [u(b_n, y) - u(a, y)]\, dy\right| = |\alpha(R\times H)|,$$

si ha:

$$|\alpha(R\times H)| \leqq \int_H g(y)\, dy,$$

donde l'asserto.

Esiste allora una funzione di y: $f(b, y)$, misurabile e sommabile in I_y, tale che:

$$\alpha(R\times H)\begin{cases} = \int_H f(b, y)\, dy & b > a \\ = -\int_H f(b, y)\, dy & b < a. \end{cases}$$

Poniamo, per $(t, y)\in I$:

$$\bar u(t, y)\begin{cases} = u(t, y) & t\in I_t - N_k \\ = f(t, y) + u(a, y) & t\in I_t \frown N_k. \end{cases}$$

Si ha allora, per ogni intervallo $J\in\{J\}$:

$$\int_{\mathcal{F}J} \bar u\, v_t\, d\omega = \alpha(J).$$

L'unicità della misura che gode della proprietà ora dimostrata per α segue immediatamente dal teor. III.

VI. *Sia $u(P)$ una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$. Se u è $[VL]_{\alpha^{(r)}, k}$ essa è anche $[VL]_{\alpha^{(1)}, k}$.*

La funzione u, per ipotesi, è equivalente in I ad una funzione verificante la $1_{\alpha^{(r)}}$). Per semplificare le notazioni, supporremo che la u stessa verifichi tale condizione. Siano a e b gli estremi di I_t. Indichiamo con J_t l'intervallo di $S_t^{(1)}$ che ha come estremo inferiore il punto a e come estremo superiore il punto t (ove $a < t \leq b$) e con $\{J_t\}$ la famiglia di tutti questi intervalli; denotiamo inoltre con $\{J_y\}$ la famiglia degli intervalli di $S_y^{(r-1)}$ contenuti in I_y e con $\{J\}$ la famiglia degli intervalli $J = J_t\times J_y$, ove $J_t\in\{J_t\}$ e $J_y\in\{J_y\}$.

Diciamo $p_{\Phi_{\alpha^{(r)}}}(J)$ e $q_{\Phi_{\alpha^{(r)}}}(J)$ rispettivamente la variazione positiva e negativa di $\Phi_{\alpha^{(r)}}$ in J[23].

Poniamo:

$$\Phi(t, J_y)\begin{cases} = \Phi_{\alpha^{(r)}}(J) & \text{se } a < t \leqq b \\ = 0 & \text{se } t = a \end{cases}$$

$$V(t, J_y)\begin{cases} = V_{\Phi_{\alpha^{(r)}}}(J) & \text{se } a < t \leqq b \\ = 0 & \text{se } t = a \end{cases}$$

$$p(t, J_y)\begin{cases} = p_{\Phi_{\alpha^{(r)}}}(J) & \text{se } a < t \leqq b \\ = 0 & \text{se } t = a \end{cases}$$

$$q(t, J_y)\begin{cases} = q_{\Phi_{\alpha^{(r)}}}(J) & \text{se } a < t \leqq b \\ = 0 & \text{se } t = a. \end{cases}$$

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[23] Dato che $\Phi_{\alpha^{(r)}}$ è una funzione additiva e VL, le sue variazioni positiva e negativa sone definite dalle relazioni:

$$p_{\Phi_{\alpha^{(r)}}}(J) = \tfrac{1}{2}[\Phi_{\alpha^{(r)}}(J) + V_{\Phi_{\alpha^{(r)}}}(J)], \quad q_{\Phi_{\alpha^{(r)}}}(J) = \tfrac{1}{2}[V_{\Phi_{\alpha^{(r)}}}(J) - \Phi_{\alpha^{(r)}}(J)].$$

Si ha, per ogni $t \in \bar{I}_t$:

(6) $$p(t, J_y) = \frac{1}{2}[\Phi(t, J_y) + V(t, J_y)], \quad q(t, J_y) = \frac{1}{2}[V(t, J_y) - \Phi(t, J_y)].$$

La funzione di J_y: $\Phi(t, J_y)$ è assolutamente continua in $\{J_y\}$ per ogni fissato t in $\bar{I}_t$, come segue dalla sua stessa definizione; $V(t, J_y)$ è funzione assolutamente continua di J_y, come risulta dai teoremi precedenti. Dalle (6) si trae allora che, per ogni $t \in \bar{I}_t$, le $p(t, J_y)$, $q(t, J_y)$ sono funzioni assolutamente continue in $\{J_y\}$.

Dato che $p_{\Phi_{\alpha(r)}}(J)$ è additiva e non negativa, per ogni $J_y \in \{J_y\}$, la $p(t, J_y)$ è funzione monotona non decrescente di t in $\bar{I}_t$.

D'altra parte esiste un insieme $N \subset I_y$, di misura $(r-1)$-dimensionale nulla, tale che, indicato con $I_{y,l}$ l'intervallo quadrato di $S_y^{(r-1)}$, contenuto in I_y, avente il punto y come estremo inferiore ed il lato di lunghezza l, per ogni $y \in I_y - N$ riesce:

$$\lim_{n \to \infty} n^{r-1} p(b, I_{y,\frac{1}{n}}) < +\infty.$$

Si ha allora, per $t \in \bar{I}_t$, $y \in I_y - N$:

$$\max_{n\to\infty} \lim n^{r-1} p(t, I_{y,\frac{1}{n}}) \leqq \lim_{n \to \infty} n^{r-1} p(b, I_{y,\frac{1}{n}}) < +\infty.$$

Per $t \in \bar{I}_t$ poniamo:

$$\mathscr{P}(t, y) \begin{cases} = \max\lim\limits_{n\to\infty} n^{r-1} p(t, I_{y,\frac{1}{n}}) & \text{se } y \in I_y - N \\ = 0 & \text{se } y \in N. \end{cases}$$

La $\mathscr{P}(t, y)$, per ogni $y \in I_y$, è funzione di t monotona non decrescente in $\bar{I}_t$: infatti, se $y \in I_y - N$ e $t' > t$, si ha: $p(t, I_{y,\frac{1}{n}}) \leqq p(t', I_{y,\frac{1}{n}})$, donde:

$$\max_{n\to\infty}\lim n^{r-1} p(t, I_{y,\frac{1}{n}}) \leqq \max_{n\to\infty}\lim n^{r-1} p(t', I_{y,\frac{1}{n}}).$$

Inoltre $\mathscr{P}(t, y)$ è non negativa dato che tale è $p(t, J_y)$. Poichè $p(t, J_y)$ è funzione assolutamente continua di J_y, per ogni $t \in \bar{I}_t$, si ha:

(7) $$p(t, J_y) = \int_{J_y} \mathscr{P}(t, y)\, dy \qquad \begin{cases} t \in \bar{I}_t \\ J_y \in \{J_y\}. \end{cases}$$

Facciamo vedere che $\mathscr{P}(t, y)$ è misurabile in I. Cominciamo, per questo, con l'osservare che la funzione di t: $p(t, I_y)$, essendo monotona non decrescente in $\bar{I}_t$, può essere discontinua soltanto in un numero finito o, al più, in un'infinità numerabile di punti di $\bar{I}_t$. Indichiamo con N_t l'insieme di questi punti. È evidente che anche la funzione di t: $p(t, J_y)$, fissato comunque J_y in $\{J_y\}$, può essere discontinua soltanto nei punti di N_t. Poniamo $\mathscr{P}_l(t, y) = l^{1-r} p(t, I_{y,l})$. La $\mathscr{P}_l(t, y)$ è funzione di y continua in I_y uniformemente rispetto a t, come segue subito dalla diseguaglianza:

$$p(t, J_y) \leqq \int_{J_y} \mathscr{P}(b, y)\, dy \qquad t \in \bar{I}_t.$$

Ne viene che, indicato con N' l'insieme contenuto in I e costituito dai punti (t, y) per ognuno dei quali si ha: $t \in N_t$, la funzione $\mathscr{P}_l(t, y)$ è continua in $I - N'$. Poichè N' ha misura r-dimensionale nulla, si ha che $\mathscr{P}_l(t, y)$ è quasi continua

e quindi misurabile in I. Poichè riesce:

$$\mathscr{P}(t,y)\begin{cases} = \max\lim\limits_{n\to\infty} \mathscr{P}_{\frac{1}{n}}(t,y) & y\in I_y - N \\ = 0 & y\in N, \end{cases}$$

resta provata la misurabilità di $\mathscr{P}(t, y)$ in I. Ripetendo per $q_{\Phi_{\alpha(r)}}(J)$ le considerazioni ora svolte per $p_{\Phi_{\alpha(r)}}(J)$, si perviene a dimostrare l'esistenza di una funzione $\mathscr{Q}(t, y)$, definita per $t\in \bar{I}_t$ e $y\in I_y$, misurabile in I, la quale, per ogni $y\in I_y$, è funzione monotona non decrescente di t in $\bar{I}_t$, è nulla in a, è funzione di y misurabile e sommabile in I_y per ogni $t\in \bar{I}_t$ ed è tale che:

(8) $$q(t, J_y) = \int_{J_y} \mathscr{Q}(t,y)\,dy \qquad \begin{cases} t\in \bar{I}_t \\ J_y\in\{J_y\}. \end{cases}$$

Per ogni $J\in\{J\}$ si ha:

$$\Phi_{\alpha(r)}(J) = p_{\Phi_{\alpha(r)}}(J) - q_{\Phi_{\alpha(r)}}(J) = p(t, J_y) - q(t, J_y),$$

cioè, per (7) ed (8):

$$\Phi_{\alpha(r)}(J) \equiv \int_{J_y} [u(t,y) - u(a,y)]\,dy = \int_{J_y} [\mathscr{P}(t,y) - \mathscr{Q}(t,y)]\,dy \qquad \begin{cases} t\in \bar{I}_t \\ J_y\in\{J_y\}. \end{cases}$$

Ne viene l'esistenza di una funzione $w(t, y)$ definita in $\bar{I}$, ivi misurabile, equivalente ad u in I, tale che, per $t\in \bar{I}_t$, $y\in I_y$ si abbia:

$$w(t,y) - u(a,y) = \mathscr{P}(t,y) - \mathscr{Q}(t,y).$$

Pertanto la funzione $w(t, y)$, per ogni $y\in I_y$, è VL rispetto a t in I_t e per la sua variazione totale $V(y)$ si ha $V(y) \leqq \mathscr{P}(b, y) + \mathscr{Q}(b, y)$. La funzione $\mathscr{P}(b, y) + \mathscr{Q}(b, y)$ è misurabile e sommabile in I_y. Ne viene che la funzione w, equivalente ad u in I, soddisfa le $1_{\alpha(1)}$), $2_{\alpha(1)}$) e pertanto u è $[VL]_{\alpha(1),\,k}$.

Così il teorema è completamente dimostrato.

VII. *Siano h un intero tale che $1\leqq h< r$, k un intero tale che $1\leqq k\leqq r$, $u(P)$ una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$. Se u è una funzione $[VL]_{\alpha(h),\,k}$ essa è anche $[VL]_{\alpha(r),\,k}$.*

Considereremo il caso $h>1$. Per $h=1$ la dimostrazione è perfettamente analoga a quella che esporremo ora.

Supponiamo, per semplicità, che la u stessa verifichi le $1_{\alpha(h)}$), $2_{\alpha(h)}$).

Sia N un insieme contenuto in I_t, di misura lineare nulla, tale che, per $t\in I_t - N$, la funzione di $(y,\eta): u(t, y, \eta)$ sia misurabile e sommabile in $I_y\times I_\eta$.

Indichiamo con $\{J\}$ la famiglia degli intervalli J di $S^{(r)}$ contenuti in I, ognuno dei quali è tale che J_t non ha gli estremi contenuti in N. Per ogni $J\in\{J\}$ risulta allora definita la funzione $\Phi_{\alpha(r)}(J) = \int_{\mathscr{F}J} u(t, y, \eta)\, v_t\, d\omega$. Mostriamo che $\Phi_{\alpha(r)}(J)$ è VL in $\{J\}$. Sia J un intervallo contenuto in $\{J\}$. Diciamo δ una arbitraria decomposizione di J: $J = J^1 + \cdots + J^p$ e poniamo $\sigma(\delta) = \sum\limits_{i=1}^{p} |\Phi_{\alpha(r)}(J^i)|$.

Consideriamo inoltre una decomposizione di J_ξ: $J_\xi = I_\xi^1 + \cdots + I_\xi^m$ e una decomposizione di $J_\eta = I_\eta^1 + \cdots + I_\eta^n$. Riesce: $J = \bigcup\limits_{i=1}^{m} \bigcup\limits_{s=1}^{n} I_\xi^i \times I_\eta^s$. Indichiamo con δ' una tale decomposizione di J. Fissata arbitrariamente una decomposizione δ

di J, può ovviamente sceglierSene una, δ', tale che $\sigma(\delta)\leqq\sigma(\delta')$. Si ha d'altra parte:

$$\sigma(\delta') = \sum_{i=1}^{m}\sum_{s=1}^{n}|\Phi_{\alpha^{(r)}}(I_\xi^i\times I_\eta^s)| = \sum_{i=1}^{m}\sum_{s=1}^{n}\left|\int_{\mathscr{I}(I_\xi^i\times I_\eta^s)} u(t,y,\eta)\, v_t\, d\omega\right|$$
$$= \sum_{i=1}^{m}\sum_{s=1}^{n}\left|\int_{I_\eta^s}\Phi_{\alpha^{(h)}}(I_\xi^i,\eta)\,d\eta\right| \leq \int_{J_\eta}\sum_{i=1}^{m}|\Phi_{\alpha^{(h)}}(I_\xi^i,\eta)|\,d\eta \leq \int_{J_\eta} g_{\alpha^{(h)}}(\eta)\,d\eta .$$

È così provato che $\Phi_{\alpha^{(r)}}(J)$ è VL in $\{J\}$. Allora, con ragionamenti perfettamente analoghi a quelli dei teor. III e V si prova l'esistenza di una funzione $\bar u$, equivalente ad u in I e verificante la $1_{\alpha^{(r)}}$).

VIII. *Se u è una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$, ivi $[VL]_{\alpha'^{(h')},k}$, indicata con $\{J\}$ la famiglia degli intervalli di $S^{(r)}$ contenuti in I, esiste una funzione w equivalente ad u in I la quale, scelti comunque l'intero h e la combinazione $\alpha^{(h)}$, verifica le $1_{\alpha^{(h)}}$), $2_{\alpha^{(h)}}$) ed è tale che, per ogni fissato η in I_η la*

$$\Phi_{\alpha^{(h)}}(J_\xi,\eta) = \int_{\mathscr{I}J_\xi} w\, v_t\, d\omega$$

è una misura; inoltre la corrispondente variazione totale: $V_{\Phi_{\alpha^{(h)}}}(I_\xi,\eta)$ è misurabile e sommabile in I_η e riesce quasi ovunque:

$$\int_{I_y} V_{\Phi_{\alpha^{(1)}}}(I_t,y,\eta)\,dy = V_{\Phi_{\alpha^{(h)}}}(I_\xi,\eta) = \mathscr{V}'(\eta),$$

ove con $\mathscr{V}'(\eta)$ si è indicata la derivata della funzione dell'intervallo J_η:

$$\mathscr{V}(J_\eta) = V_{\Phi_{\alpha^{(r)}}}(I_\xi\times J_\eta).$$

Per i teoremi VI e VII possiamo supporre che la u stessa verifichi le $1_{\alpha^{(1)}}$), $2_{\alpha^{(1)}}$).

Per i teoremi I e II esiste una funzione $\tilde u$, equivalente ad u in I, la quale, per ogni fissato τ in I_τ, è continua rispetto ad $x_k=t$ nei punti estremi di I_t ed è continua a sinistra in ogni punto interno di I_t. Inoltre la sua variazione totale $\tilde V(\tau)$ è misurabile e sommabile in I_τ.

Siano a e b gli estremi di I_t.

È facile vedere l'esistenza di un insieme N_τ contenuto in $\bar I_\tau$, di misura $(r-1)$-dimensionale nulla verificante le seguenti condizioni:

1°) indicata con $\{\Gamma\}$ la totalità delle varietà lineari ognuna delle quali è definita da equazioni del tipo $x_{b_1}=\text{cost.}, \ldots, x_{b_{r-h}}=\text{cost.}$ (ove $b_1,\ldots,b_{r-h}$ è un'arbitraria combinazione dei numeri $1, 2, \ldots, k-1, k+1, \ldots, r$) e posto:

$$\bar V(\tau)\begin{cases} =\tilde V(\tau) & \tau\in\bar I_\tau - N_\tau \\ =0 & \tau\in N_\tau, \end{cases}$$

la $\bar V(\tau)$ risulta misurabile e sommabile su ogni intersezione di Γ con l'insieme definito da: $t=a$, $\tau\in I_\tau$ al variare comunque di Γ in $\{\Gamma\}$,

2°) la funzione così definita in $\bar I$:

$$\bar u(t,\tau)\begin{cases} =\tilde u(t,\tau) & \tau\in\bar I_\tau - N_\tau \\ =0 & \tau\in N_\tau \end{cases}$$

è misurabile e sommabile su ogni intersezione $\Gamma\cap\bar I$ al variare comunque di Γ in $\{\Gamma\}$.

Sia h un intero compreso tra 1 ed r ed $\alpha^{(h)}$ una combinazione di classe h dei numeri $1, 2, \ldots, k-1, k+1, \ldots, r$. È facile constatare che la $\bar{u}(t, \tau)$ verifica la $1_{\alpha^{(h)}}$). Siano η un punto di I_η, δ una decomposizione di I_t relativa ai punti $\alpha_0, \alpha_1, \ldots, \alpha_n$. Si ha:

$$V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta) = \underset{\{\delta\}}{\text{estr. sup.}} \sum_{s=1}^{n} \int_{I_y} |\bar{u}(\alpha_s, y, \eta) - \bar{u}(\alpha_{s-1}, y, \eta)| \, dy .$$

La dimostrazione di questa relazione è analoga a quella con cui si è provata la (4).

Poichè si ha

$$\sum_{s=1}^{n} |\bar{u}(\alpha_s, y, \eta) - \bar{u}(\alpha_{s-1}, y, \eta)| \leqq \bar{V}(y, \eta)$$

e:

$$\underset{\{\delta\}}{\text{estr. sup.}} \sum_{s=1}^{n} |\bar{u}(\alpha_s, y, \eta) - \bar{u}(\alpha_{s-1}, y, \eta)| = V_{\bar{\Phi}_{\alpha^{(1)}}}(y, \eta) ,$$

riesce:

$$\int_{I_y} V_{\bar{\Phi}_{\alpha^{(1)}}}(y, \eta) \, dy = V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta) .$$

Resta così anche provato che $V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta)$ è funzione di η misurabile in I_η.

Inoltre, da (4), si trae:

$$\underset{\{\delta\}}{\text{estr. sup.}} \sum_{s=1}^{n} \int_{J_\eta} \left[\int_{I_y} |\bar{u}(\alpha_s, y, \eta) - \bar{u}(\alpha_{s-1}, y, \eta)| \, dy \right] d\eta = \mathscr{V}(J_\eta) .$$

Poichè:

$$\underset{\{\delta\}}{\text{estr. sup.}} \sum_{s=1}^{n} \int_{I_y} |\bar{u}(\alpha_s, y, \eta) - \bar{u}(\alpha_{s-1}, y, \eta)| \, dy = V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta) ,$$

tenuto conto della assoluta continuità di $\mathscr{V}(J_\eta)$, si ha:

$$\int_{J_\eta} V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta) \, d\eta = \mathscr{V}(J_\eta) = \int_{J_\eta} \mathscr{V}'(\eta) \, d\eta .$$

Sia N_η l'insieme contenuto in I_η, di misura $(r-h)$-dimensionale nulla, tale che, per $\eta \in I_\eta - N_\eta$, riesce:

$$V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta) = \mathscr{V}'(\eta) .$$

Indichiamo con N l'insieme, di misura r-dimensionale nulla, costituito da tutti e soli i punti che godono della seguente proprietà: se $\alpha^{(h)}$ è una combinazione di classe h dei numeri $1, 2, \ldots, r$, per ogni punto (ξ, η), contenuto in N riesce $\eta \in N_\eta$.

La funzione $w(P)$ così definita in $\bar{I}$:

$$w(P) \begin{cases} = \bar{u}(P) & \text{se } P \in \bar{I} - N \\ = 0 & \text{se } P \in N \end{cases}$$

gode di tutte le proprietà ora dimostrate per $\bar{u}$ ed inoltre è tale che, fissato comunque $\alpha^{(h)}$, riesce quasi ovunque:

$$V_{\Phi_{\alpha^{(h)}}}(I_\xi, \eta) = \mathscr{V}'(\eta) .$$

Resta soltanto da provare che, per ogni $\alpha^{(h)}$, la $\Phi_{\alpha^{(h)}}(J_\xi, \eta)$ è una misura nella famiglia $\{J_\xi\}$ degli intervalli di $S_\xi^{(h)}$ che sono contenuti in I_ξ.

Dai teoremi IV e VI discende che, per ogni $\alpha^{(h)}$, esistono una misura $\mu_{\alpha^{(h)}}(J_\xi, \eta)$, definita, per ogni η, nella famiglia $\{J_\xi\}$ e un insieme $N_t^{\alpha^{(h)}}(\eta)$ verificanti le seguenti proprietà:

1) $N_t^{\alpha^{(h)}}(\eta)$ è contenuto in I_t ed ha misura lineare nulla;

2) per ogni intervallo J_ξ: $y \in J_y$, $a \leqq t < b$ con $a \in I_t - N_t^{\alpha^{(h)}}(\eta)$, $b \in I_t - N_t^{\alpha^{(h)}}(\eta)$, riesce: $\mu_{\alpha^{(h)}}(J_\xi) = \Phi_{\alpha^{(h)}}(J_\xi, \eta)$.

Sia ora J_ξ un intervallo di $\{J_\xi\}$ definito dalle condizioni: $y \in J_y$ (ove J_y è un intervallo di $S_y^{(h-1)}$ contenuto in I_y) e $a' \leqq t < b'$, con $a' \in I_t - N_t^{\alpha^{(h)}}(\eta)$, $b' \in I_t$. Se $\{t_n\}$ è una successione di punti di I_t tale che $a' < t_n < b'$, $t_n \in I_t - N_t^{\alpha^{(h)}}(\eta)$, $\lim\limits_{n\to\infty} t_n = b'$ e se $J_\xi^{(n)}$ è l'intervallo definito da: $y \in J_y$, $a' \leqq y < t_n$ si ha:

$$\lim_{n\to\infty} \mu_{\alpha^{(h)}}(J_\xi^{(n)}, \eta) = \mu_{\alpha^{(h)}}(J_\xi, \eta) \qquad \text{e} \qquad \lim_{n\to\infty} \Phi_{\alpha^{(h)}}(J_\xi^{(n)}, \eta) = \Phi_{\alpha^{(h)}}(J_\xi, \eta).$$

Ne viene la coincidenza di $\mu_{\alpha^{(h)}}$ con $\Phi_{\alpha^{(h)}}$, su tutti gli intervalli di $S^{(h)}$ contenuti in I_ξ.

In tal modo il teorema è completamente dimostrato. Sia A un campo di $S^{(r)}$. Diremo che la funzione $u(P)$, misurabile e localmente sommabile in A (cioè sommabile in ogni I tale che $\bar{I} \subset A$ è $[VL]_{\alpha^{(h)},k}$ *nel campo* A se esiste una funzione equivalente ad u in detto campo la quale verifichi le condizioni $1_{\alpha^{(h)}}$), $2_{\alpha^{(h)}}$) in corrispondenza ad ogni I tale che $\bar{I} \subset A$. Fissata una combinazione $\alpha^{(h)}$, indicheremo con A_t l'insieme di tutti i punti t_0 di $S_t^{(1)}$ in corrispondenza ad ognuno dei quali l'iperpiano d'equazione $t = t_0$ ha intersezione non vuota con A. In modo analogo definiremo gli insiemi $A_\eta, A_\xi, A_y, A_{y,\eta}$, *etc.* Denoteremo inoltre con $\mathscr{S}_A(t_0)$ l'intersezione di A con l'iperpiano d'equazione $t = t_0$. In modo analogo si definiranno gli insiemi $\mathscr{S}_A(\eta)$, $\mathscr{S}_A(\xi)$, $\mathscr{S}_A(y)$. Indicheremo infine con $\{I\}_A$ la famiglia degli intervalli di $S^{(r)}$ contenuti con il proprio involucro in A, con $\{I_\xi\}_\eta$ quella degli intervalli di $S_\xi^{(h)}$ contenuti con il proprio involucro in $\mathscr{S}_A(\eta)$ per $\eta \in A_\eta$. Analogamente si definiscono le famiglie $\{I_\xi\}_\eta$, $\{I_t\}_{y,\eta}$ *etc.*

Dimostriamo il seguente teorema:

IX. *Sia $u(P)$ una funzione definita nel campo A, ivi misurabile e localmente sommabile. Condizione necessaria e sufficiente perchè u sia $[VL]_{\alpha^{(h)},k}$ in A è che essa sia $[VL]_{\alpha^{(h)},k}$ in ogni intervallo $I \subset \{I\}_A$.*

La necessità della condizione è ovvia. Dimostriamone la sufficienza. Supponiamo $1 < h < r$. Indichiamo con $\{I\}$ una famiglia costituita da un'infinità numerabile di intervalli di $\{I\}_A$, a due a due privi di punti in comune, tali inoltre che la loro riunione coincida con A e che ogni intervallo I di $\{I\}_A$ abbia intersezione non vuota soltanto con un numero finito di intervalli di $\{I\}$. Con i teoremi VI, VII, e VIII si è fatto vedere che, se u è $[VL]_{\alpha^{(h)},k}$ in un intervallo I di $S^{(r)}$, si può costruire una funzione $\bar{u}(t, y, \eta)$ equivalente ad u in $\bar{I}$, la quale soddisfa le condizioni $1_{\alpha^{(h)}}$) e $2_{\alpha^{(h)}}$), ed inoltre, fissato t in $\bar{I}_t$, risulta misurabile e sommabile in $I_y \times I_\eta$. Applicando in ogni intervallo I di $\{I\}$ il detto procedimento, si perviene a costruire una funzione $\bar{u}$ equivalente ad u in A, la quale gode delle seguenti proprietà: scelto comunque un intervallo I di $\{I\}$, esiste una funzione w, definita in $\bar{I}$, coincidente con $\bar{u}$ in I, verificante le condizioni $1_{\alpha^{(h)}}$) e $2_{\alpha^{(h)}}$) relativamente ad I e tale che, fissato t in $\bar{I}_t$, $w(t, y, \eta)$ risulti misurabile e sommabile in $I_y \times I_\eta$; inoltre, per ogni $t \in \bar{I}_t$, la $\bar{u}(t, y, \eta)$ è essa stessa misurabile e sommabile in $I_y \times I_\eta$ e verifica la $1_{\alpha^{(h)}}$) per ogni $I \in \{I\}_A$.

Facciamo vedere che la funzione $\bar{u}$ verifica le condizioni $1_{\alpha^{(h)}})$, $2_{\alpha^{(h)}})$ in ogni intervallo I di $\{I\}$. Sia $I \in \{I\}$. Indichiamo con a e b gli estremi di I_t e con $\{J_\xi\}$ la famiglia degli intervalli di $S^{(h)}$ ognuno dei quali è contenuto in I_ξ. Poniamo $\Phi_{\alpha^{(h)}}(J_\xi, \eta) = \int_{\mathscr{F} J_\xi} w(\xi, \eta)\, \nu_t\, d\omega$ ed indichiamo con $g_{\alpha^{(h)}}(\eta)$ una funzione misurabile e sommabile in I_η tale che $V_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta) \leqq g_{\alpha^{(h)}}(\eta)$. Poniamo, per ogni $\eta \in I_\eta$:

$$\bar{\Phi}_{\alpha^{(h)}}(J_\xi, \eta) = \int_{\mathscr{F} J_\xi} \bar{u}(\xi, \eta)\, \nu_t\, d\omega .$$

Se J_ξ è un intervallo appartenente a $\{J_\xi\}$ lacui frontiera non ha alcun punto nell'iperpiano d'equazione $t = b$, riesce: $\Phi_{\alpha^{(h)}}(J_\xi, \eta) = \bar{\Phi}_{\alpha^{(h)}}(J_\xi, \eta)$. Indichiamo con δ un'arbitraria decomposizione I_t in intervalli mediante i punti $\alpha_0 \equiv a < \alpha_1 < \cdots < \alpha_n \equiv b$ ed indichiamo con J_ξ^s l'intervallo di $S_\xi^{(h)}$ definito dalle condizioni: $\alpha_{s-1} \leqq t < \alpha_s$, $y \in I_y$. Poniamo

$$\sigma(\delta) = \sum_{s=1}^{n} \int_{I_y} |\bar{u}(\alpha_s, y, \eta) - \bar{u}(\alpha_{s-1}, y, \eta)|\, dy .$$

Si ha:

$$\sigma(\delta) \leqq \sum_{s=1}^{n} \int_{I_y} |w(\alpha_s, y, \eta) - w(\alpha_{s-1}, y, \eta)|\, dy + \int_{I_y} |w(b, y, \eta) - \bar{u}(b, y, \eta)|\, dy .$$

Da tale diseguaglianza si trae:

$$V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta) \leqq V_{\Phi_{\alpha^{(h)}}}(I_\xi, \eta) + \int_{I_y} |w(b, y, \eta) - \bar{u}(b, y, \eta)|\, dy$$

e quindi:

$$V_{\bar{\Phi}_{\alpha^{(h)}}}(I_\xi, \eta) \leqq g_{\alpha^{(h)}}(\eta) + \int_{I_y} |w(b, y, \eta) - \bar{u}(b, y, \eta)|\, dy .$$

Poichè il secondo membro di tale diseguaglianza è una funzione misurabile e sommabile in I_η, resta provato che $\bar{u}$ verifica le $1_{\alpha^{(h)}})$, $2_{\alpha^{(h)}})$ relativamente all'intervallo I di $\{I\}$. È allora evidente che $\bar{u}$ verifica tali condizioni anche in relazione a qualsiasi intervallo $I \in \{I\}_A$.

X. *Se u è una funzione misurabile e localmente sommabile nel campo A ed ivi $[VL]_{\alpha^{(h)}, k}$, esiste una funzione w equivalente ad u in A la quale, fissati comunque l'intero h' e la combinazione $\alpha'^{(h')}$, verifica le $1_{\alpha'^{(h')}})$ e $2_{\alpha'^{(h')}})$ in corrispondenza ad ogni $I \in \{I\}_A$, inoltre è tale che, per ogni fissato η in A_η, la $\Phi_{\alpha'^{(h')}}(I_\xi, \eta)$ è una misura in $\{I_\xi\}_\eta$ e, fissato comunque I in $\{I\}_A$, quasi ovunque in I_η riesce:*

$$\int_{I_y} V_{\Phi_{\alpha^{(1)}}}(I_t, y, \eta)\, dy = \mathscr{V}'(\eta) = V_{\Phi_{\alpha'^{(h')}}}(I_\xi, \eta),$$

ove con $\mathscr{V}'(\eta)$ si è indicata la derivata della funzione di J_η: $V_{\Phi_{\alpha^{(r)}}}(I_\xi \times J_\eta)$ nella famiglia degli intervalli J_η in corrispondenza ad ognuno dei quali riesce: $I_\xi \times J_\eta \in \{I\}_A$.

Infatti, per i teorr. VII, IX, la u è $[VL]_{\alpha^{(1)}, k}$ in A. Possiamo allora supporre che la u verifichi le $1_{\alpha^{(1)}})$ e $2_{\alpha^{(1)}})$ in corrispondenza ad ogni $I \in \{I\}_A$. Indichiamo con $\{I^{(s)}\}$ una successione di intervalli di $\{I\}_A$, a due a due privi di punti in comune, tale che: $A = \bigcup_{s=1}^{\infty} I^{(s)}$ e tale che ogni intervallo $I \in \{I\}_A$ abbia prodotto non vuoto soltanto con un numero finito di insiemi di $\{I^{(s)}\}$. Per ogni s indichiamo con $u^{(s)}$ la funzione definita in $\overline{I^{(s)}}$ e dedotta da u con il procedimento indicato

nella dimostrazione del teor. II. Siano $(a_1^{(s)}, \ldots, a_r^{(s)})$ e $(b_1^{(s)}, \ldots, b_r^{(s)})$ i punti estremi di $I^{(s)}$. Consideriamo la funzione $\tilde{u}$ la quale coincide con $u^{(s)}$ per ogni $P \equiv (x_1, \ldots, x_r)$ tale che $a_i^{(s)} < x_i \leqq b_i^{(s)}$. La funzione $\tilde{u}$, per ogni $(y, \eta) \in A_{y,\eta}$, riesce continua a sinistra rispetto a t in ogni punto di $\mathscr{S}_A(y, \eta)$. Sia $w^{(s)}$ la funzione definita in $I^{(s)}$ e dedotta da $\tilde{u}$ con il procedimento indicato nella dimostrazione del teor. VIII. La funzione w che coincide con $w^{(s)}$ in $I^{(s)}$ $(s = 1, 2, \ldots)$ soddisfa le condizioni richieste dal teorema, come è immediato verificare.

Vogliamo ora occuparci della definizione di derivata rispetto ad x_k per una funzione che sia $[VL]_{\alpha^{(h)}, k}$ e dell'equivalenza delle derivate $\left[\frac{\partial u}{\partial x_k}\right]_{\alpha^{(h)}, k}$ al variare in tutti i modi possibili della combinazione $\alpha^{(h)}$.

Facciamo innanzitutto vedere che:

XI. *Se $u(P)$ è una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$, ivi misurabile e sommabile, e verificante la condizione $1_{\alpha^{(h)}}$), si ha che la derivata $\Phi'_{\alpha^{(h)}}(P)$ della corrispondente funzione d'intervallo $\Phi_{\alpha^{(h)}}(I_\xi, \eta)$ è misurabile in I. Inoltre, se $\bar{u}$ è un'altra funzione verificante le stesse ipotesi fatte per u e tale che $u = \bar{u}$ quasi ovunque in I, si ha che la derivata $\bar{\Phi}'_{\alpha^{(h)}}$ della corrispondente funzione d'intervallo $\bar{\Phi}_{\alpha^{(h)}}(I_\xi, \eta)$ è equivalente a $\Phi'_{\alpha^{(h)}}(P)$ in I.*

Siano a e b gli estremi di I_t: Indichiamo con $I_{y,l}$ l'intersezione di I_y con l'intervallo quadrato di $S_y^{(h-1)}$ che ha l'estremo inferiore nel punto y di I_y ed il lato di lunghezza l. La funzione $\varphi_l(P) \equiv \varphi_l(t, y, \eta) = \int\limits_{I_{y,l}} u(t, y', \eta)\, dy'$ è misurabile in I. Basta evidentemente limitarsi a dimostrare ciò nel caso che $u(P)$ sia non negativa. In tale ipotesi esiste, com'è noto, una successione non decrescente $\{u_n(t, y, \eta)\}$ di funzioni misurabili e costanti a tratti, convergente ad $u(P)$ in I. Siano $L_n^{(1)}, \ldots, L_n^{(m_n)}$ gli insiemi in ognuno dei quali u_n è costante. Diciamo $\Psi_n^i(P)$ $(i = 1, \ldots, m_n)$ la funzione nulla in $I - L_n^{(i)}$ ed eguale ad 1 in $L_n^{(i)}$; se $a_n^{(i)}$ è il valore assunto da u_n nei punti di $L_n^{(i)}$, si ha: $u_n(P) = \sum\limits_{i=1}^{m_n} a_n^{(i)} \Psi_n^{(i)}(P)$. La funzione $\int\limits_{I_{y,l}} \Psi_n^{(i)}(t, y', \eta)\, dy'$ è misurabile in I; ne viene che tale è anche la funzione di P: $\int\limits_{I_{y,l}} u_n(t, y', \eta)\, dy'$. Poichè $\int\limits_{I_{y,l}} u(t, y', \eta)\, dy' = \lim\limits_{n\to\infty} \int\limits_{I_{y,l}} u_n(t, y', \eta)\, dy'$, si ha l'asserto. Da ciò si trae che la funzione di P: $\varphi_l(P)$ è misurabile in I. Sia E l'insieme dei punti di I nei quali non esiste finito $\lim\limits_{n\to\infty} n^h \left| \varphi_{\frac{1}{n}}\left(t + \frac{1}{u}, \tau\right) \quad \varphi_{\frac{1}{n}}(t, \tau) \right|$. L'insieme E è misurabile. Ne viene che la funzione $\Phi'_{\alpha^{(h)}}(P)$, la quale coincide con tal limite nei punti di $I - E$ e vale zero nei punti di E, è misurabile in I. Sia N un insieme contenuto in I_η, di misura $(r-h)$-dimensionale nulla, tale che, per $\eta \in I_\eta - N$, l'insieme dei punti di I_ξ in ognuno dei quali $u(\xi, \eta) \neq \bar{u}(\xi, \eta)$ abbia misura h-dimensionale nulla. Se η è un punto di $I_\eta - N$, indicheremo con $N(\eta)$ un insieme contenuto in I_t, di misura lineare nulla, tale che, per $t \in I_t - N(\eta)$, l'insieme dei punti di I_y nei quali $u(t, y, \eta) \neq \bar{u}(t, y, \eta)$ abbia misura $(h-1)$-dimensionale nulla. Sia $\eta \in I_\eta - N$; indichiamo con $\{J_\xi\}_\eta$ la famiglia degli intervalli J_ξ di $S_\xi^{(h)}$ ognuno dei quali è definito dalle condizioni $y \in J_y$, ove J_y è un intervallo di $S_y^{(h-1)}$ tale che $\bar{J}_y \subset I_y - \mathscr{J} I_y$, $\alpha \leqq t < \beta$ con $\alpha \in I_t - (N(\eta) + a)$, $\beta \in I_t - (N(\eta) + b)$. È immediato constatare che la famiglia $\{J_\xi\}_\eta$ costituisce un semianello tale che la famiglia

d'insiemi totalmente additiva minima che lo contiene è quella dei boreliani aventi il proprio involucro in $I_\xi - \mathcal{J} I_\xi$.

Si ha, ovviamente, per ogni $\eta \in I_\eta - N$ e per ogni $J_\xi \in \{J_\xi\}_\eta$:

$$\Phi_{\alpha^{(h)}}(J_\xi, \eta) = \overline{\Phi}_{\alpha^{(h)}}(J_\xi, \eta). \tag{9}$$

Consideriamo la decomposizione di LEBESGUE di $\Phi_{\alpha^{(h)}}$ relativa alla famiglia $\{J_\xi\}_\eta$:

$$\Phi_{\alpha^{(h)}}(J_\xi, \eta) = \Phi^*_{\alpha^{(h)}}(J_\xi, \eta) + \int_{J_\xi} \Phi'_{\alpha^{(h)}}(\xi, \eta)\, d\xi, \tag{10}$$

ove $\Phi^*_{\alpha^{(h)}}(J_\xi, \eta)$ è la componente singolare di $\Phi_{\alpha^{(h)}}(J_\xi, \eta)$. Da (9) e (10) e dalla analoga relazione scritta per $\overline{\Phi}_{\alpha^{(h)}}$ si trae, in particolare, per ogni $\eta \in I_\eta - N$, $J_\xi \in \{J_\xi\}_\eta$:

$$\int_{J_\xi} \Phi'_{\alpha^{(h)}}(\xi, \eta)\, d\xi = \int_{J_\xi} \overline{\Phi}'_{\alpha^{(h)}}(\xi, \eta)\, d\xi.$$

Ne viene che, fissato η in $I_\eta - N$, per ogni boreliano B_ξ di $S^{(h)}_\xi$ contenuto, con il proprio involucro, in I_ξ, deve essere:

$$\int_{B_\xi} \Phi'_{\alpha^{(h)}}(\xi, \eta)\, d\xi = \int_{B_\xi} \overline{\Phi}'_{\alpha^{(h)}}(\xi, \eta)\, d\xi.$$

Se ne conclude, che, per $\eta \in I_\eta - N$, quasi ovunque in I_ξ riesce: $\Phi'_{\alpha^{(h)}}(\xi, \eta) = \overline{\Phi}'_{\alpha^{(h)}}(\xi, \eta)$. Da ciò segue l'asserto. Con dimostrazione analoga a questa si provano i teoremi:

XII. *Se $u(P)$ è una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$, ivi misurabile, sommabile e verificante la condizione $1_{\alpha^{(1)}})$, allora la derivata $\dfrac{\partial u}{\partial x_k}$ è misurabile in I. Se $\bar{u}(P)$ è una funzione verificante le stesse ipotesi fatte per u, ed equivalente ad u in I, si ha che $\dfrac{\partial u}{\partial x_k} = \dfrac{\partial \bar{u}}{\partial x_k}$ quasi ovunque in I.*

XIII. *Se $u(P)$ ed $\bar{u}(P)$ sono due funzioni definite nell'involucro dell'intervallo I di $S^{(r)}$, ivi misurabili, sommabili e verificanti la condizione $1_{\alpha^{(r)}})$, e se $u(P)$ ed $\bar{u}(P)$ sono equivalenti in I, si ha $\Phi'_{\alpha^{(r)}}(P) = \overline{\Phi}'_{\alpha^{(r)}}(P)$ quasi ovnnque in I.*

Questi teoremi, come già si è detto, consentono di definire la derivata rispetto ad x_k: $\left[\dfrac{\partial u}{\partial x_k}\right]_{\alpha^{(h)}, k}$ di una funzione u che sia $[VL]_{\alpha^{(h)}, k}$ in un intervallo I di $S^{(r)}$.

Dimostriamo che:

XIV. *Se $u(P)$ è una funzione definita nell'involucro dell'intervallo I di $S^{(r)}$ la quale sia ivi $[VL]_{\alpha^{(h)}, k}$ esiste una funzione $\bar{u}$ equivalente ad u in I, la quale gode della seguente proprietà: per ogni $(y, \eta) \in I_y \times I_\eta$ è VL rispetto a t in I_t; la sua derivata $\dfrac{\partial \bar{u}}{\partial x_k}$ è misurabile in I e riesce: $\dfrac{\partial \bar{u}}{\partial x_k} = \left[\dfrac{\partial u}{\partial x_k}\right]_{\alpha^{(h)}, k}$.*

Per $h = 1$ il teorema è evidente. Supponiamo $1 < h < r$. Siano a e b gli estremi di I_t. Denotiamo con $\{J_\xi\}$ la famiglia degli intervalli di $S^{(h)}_\xi$ contenuti in I_ξ, per ognuno dei quali si ha: $J_\xi - J_t \times J_y$ ove J_t è l'intervallo di $S^{(1)}_t$ che ha estremo inferiore a ed estremo superiore t, ed ove J_y è un intervallo di $S^{(h-1)}_y$ contenuto in I_y. Consideriamo la funzione w, equivalente ad u in I e dedotta dalla u con il procedimento esposto nella dimostrazione del teor. VIII. Denotiamo con $p_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta)$, $q_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta)$, $V_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta)$ rispettivamente le variazioni

positiva, negativa e totale di $\Phi_{\alpha^{(h)}}(J_\xi, \eta) = \int_{\mathscr{J} J_\xi} w(\xi, \eta)\, \nu_t \, d\omega$ in J_ξ, per ogni $\eta \in I_\eta$. Poniamo, per $\eta \in I_\eta$, $J_\xi \in \{J_\xi\}$:

$$\Phi(J_y, t, \eta) \begin{cases} = \Phi_{\alpha^{(h)}}(J_\xi, \eta), & a < t \leqq b \\ = 0 & a = 0 \end{cases}$$

$$V(J_y, t, \eta) \begin{cases} = V_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta) & a < t \leqq b \\ = 0 & t = a \end{cases}$$

$$p(J_y, t, \eta) \begin{cases} = p_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta) & a < t \leqq b \\ = 0 & t = a \end{cases}$$

$$q(J_y, t, \eta) \begin{cases} = q_{\alpha^{(h)}}(J_\xi, \eta) & a < t \leqq b \\ = 0 & t = a. \end{cases}$$

Sia $I_{y,l}$ l'intersezione di I_y con l'intervallo quadrato che ha come estremo inferiore il punto y di I_y ed il lato di lunghezza l. La funzione di P: $\Phi(I_{y,l}, t, \eta)$ è misurabile in I, come osservato nel corso della dimostrazione del teor. XI. Proviamo ora la misurabilità di $V(I_{y,l}, t, \eta)$. Innanzitutto è facile vedere che se $a < t \leqq b$, indicato con $s(t)$ il più grande intero non negativo, tale che: $a + s(t) \cdot \frac{b-a}{n} \leqq t$, riesce:

$$V(I_{y,l}, t, \eta) = \lim_{n \to \infty} \sum_{s=1}^{s(t)} \int_{I_{y,l}} |w(\alpha_s^{(n)}, y', \eta) - w(\alpha_{s-1}^{(n)}, y', \eta)|\, dy'. \tag{11}$$

Infatti, ripetendo un ragionamento analogo ad uno fatto nel corso della dimostrazione del teor. VIII, si vede che, indicato con $\{\delta_t\}$ l'insieme delle decomposizioni di $(0, t)$, si ha:

$$V(I_{y,l}, t, \eta) = \underset{\{\delta_t\}}{\text{estr. sup.}} \sum_{s=1}^{m} \int_{I_{y,l}} |w(\alpha_s, y', \eta) - w(\alpha_{s-1}, y', \eta)|\, dy'.$$

Poichè $w(t, y, \eta)$ è funzione di t continua nei punti a e b e continua a sinistra in ogni punto interno ad I_t, è facile dimostrare che

$$\begin{aligned} &\underset{\{\delta_t\}}{\text{estr. sup}} \sum_{s=1}^{m} \int_{I_{y,l}} |w(\alpha_s, y', \eta) - w(\alpha_{s-1}, y', \eta)|\, dy' \\ &\qquad = \lim_{n \to \infty} \sum_{s=1}^{s(t)} \int_{I_{y,l}} |w(\alpha_s^{(n)}, y', \eta) - w(\alpha_{s-1}^{(n)}, y', \eta)|\, dy'. \end{aligned}$$

Resta così provata la (11). La funzione

$$\varphi_n(t, y, \eta) = \sum_{s=1}^{s(t)} \int_{I_{y,l}} |w(\alpha_s^{(n)}, y', \eta) - w(\alpha_{s-1}^{(n)}, y', \eta)|\, dy',$$

per ogni fissato t in I_t, è misurabile rispetto ad (y, η) e per ogni $(y, \eta) \in I_y \times I_\eta$ è funzione di t costante a tratti (precisamente è costante in ognuno degli intervalli $\alpha_{s-1}^{(n)} \leqq t < \alpha_s^{(n)}$). Pertanto $\varphi_n(t, y, \eta)$ è misurabile in I. Dalla (11) segue che tale risulta anche la $V(I_{y,l}, t, \eta)$. Si ha per ogni $\eta \in I_\eta$, $t \in \bar{I}_t$, $J_y \in \{J_y\}$:

$$\begin{aligned} p(J_y, t, \eta) &= \tfrac{1}{2}[\Phi(J_y, t, \eta) + V(J_y, t, \eta)] \\ q(J_y, t, \eta) &= \tfrac{1}{2}[V(J_y, t, \eta) - \Phi(J_y, t, \eta)]. \end{aligned} \tag{12}$$

Per ogni $\eta \in I_\eta$, $t \in \bar{I}_t$, le funzioni di J_y: $\Phi(J_y, t, \eta)$ e $V(J_y, t, \eta)$ sono assolutamente continue in $\{J_y\}$ come segue dalla definizione stessa di Φ e dal teor. IV. Dalle (12) si trae allora che, per ogni $\eta \in I_\eta$, $t \in \bar{I}_t$, le funzioni di J_y: $p(J_y, t, \eta)$, $q(J_y, t, \eta)$ sono assolutamente continue in $\{J_y\}$.

Fissato η in I_η, consideriamo la decomposizione di LEBESGUE di $p_{\alpha^{(h)}}(J_\xi, \eta)$:

(13) $$p_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta) = p^*_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta) + \int_{J_\xi} p'_{\Phi_{\alpha^{(h)}}}(\xi, \eta)\, d\xi$$

ove $p^*_{\alpha^{(h)}}(J_\xi, \eta)$ è la componente singolare di $p_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta)$. Posto, per $\eta \in I_\eta$ e $J_\xi \in \{J_\xi\}$:

$$p^*(J_y, t, \eta) \begin{cases} = p^*_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta) & a < t \leqq b \\ = 0 & t = a, \end{cases}$$

la (13) può scriversi:

(14) $$p(J_y, t, \eta) = p^*(J_y, t, \eta) + \int_{J_y} dy \int_a^t p'_{\Phi_{\alpha^{(h)}}}(t', y, \eta)\, dt'.$$

Da questa relazione si trae che, per $\eta \in I_\eta$ e $t \in \bar{I}_t$, la $p^*(J_y, t, \eta)$ è funzione assolutamente continua in $\{J_y\}$. Dato che, per ogni $\eta \in I_\eta$, la $p^*_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta)$ è funzione di J_ξ additiva e non negativa in $\{J_\xi\}$, per $\eta \in I_\eta$, $J_y \in \{J_y\}$ la funzione di t: $p^*(J_y, t, \eta)$ è monotona non decrescente in $\bar{I}_t$. Inoltre la funzione $p^*(I_{y,l}, t, \eta)$ è misurabile in I. Infatti dalle (12) e dalla misurabilità delle funzioni $\Phi(I_{y,l}, t, \eta)$ e $V(I_{y,l}, t, \eta)$ si trae quella della funzione $p(I_{y,l}, t, \eta)$ e quindi quella della funzione $p'_{\Phi_{\alpha^{(h)}}}(t, y, \eta)$. È così provata la misurabilità di $\int_{I_{y,l}} dy' \int_a^t p'_{\Phi_{\alpha^{(h)}}}(t', y', \eta)\, dt'$ e quindi, per la (14), anche quella di $p^*(I_{y,l}, t, \eta)$. Pertanto l'insieme E dei punti di $\bar{I}$ nei quali non esiste finito $\max\lim_{n\to\infty} n^{h-1} p^*(I_{y,\frac{1}{n}}, t, \eta)$ è misurabile ed inoltre la funzione così definita in $\bar{I}$:

$$\varphi(t, y, \eta) \begin{cases} = \max\lim\limits_{n\to\infty} n^{h-1} p^*(I_{y,\frac{1}{n}}, t, \eta) & \text{in } \bar{I} - E \\ = 0 & \text{in } E \end{cases}$$

è misurabile in I.

La funzione $\varphi(t, y, \eta)$, per ogni $(y, \eta) \in I_y \times I_\eta$, è funzione di t monotona non decrescente in $\bar{I}_t$; infatti, se $t' > t$, si ha: $p^*(I_{y,\frac{1}{n}}, t, \eta) \leqq p^*(I_{y,\frac{1}{n}}, t', \eta)$, donde:

$$\max\lim_{n\to\infty} n^{h-1} p^*(I_{y,\frac{1}{n}}, t, \eta) \leqq \max\lim_{n\to\infty} n^{h-1} p^*(I_{y,\frac{1}{n}}, t', \eta).$$

Pertanto la derivata $\frac{\partial\varphi}{\partial t}$ è non negativa. Inoltre $\varphi(t, y, \eta)$ è non negativa dato che tale è $p^*_{\Phi_{\alpha^{(h)}}}(J_\xi, \eta)$. Poichè, per $\eta \in I_\eta$, $t \in \bar{I}_t$, la $p^*(J_y, t, \eta)$ è funzione di J_y assolutamente continua in $\{J_y\}$, la $\varphi(t, y, \eta)$ è funzione di y misurabile e sommabile in I_y e si ha:

$$p^*(J_y, t, \eta) = \int_{J_y} \varphi(t, y, \eta)\, dy \qquad \begin{cases} \eta \in I_\eta, & t \in \bar{I}_t \\ J_y \in \{J_y\}. \end{cases}$$

Riesce pertanto:

(15) $$p(J_y, t, \eta) = \int_{J_y} \left[\varphi(t, y, \eta) + \int_a^t p'_{\Phi_{\alpha^{(h)}}}(t', y, \eta)\, dt'\right] dy \qquad \begin{cases} \eta \in I_\eta, & t \in \bar{I}_t \\ J_y \in \{J_y\}. \end{cases}$$

Facciamo ora vedere che $\frac{\partial}{\partial t}\varphi(t, y, \eta)$ è quasi ovunque nulla in I. Infatti, se così non fosse, dato che $\frac{\partial \varphi}{\partial t}$ è sempre non negativa, esisterebbero un punto η contenuto in I_η, un numero positivo $\varrho_0(\eta)$ ed un insieme $E(\eta) \in I_\xi$, di misura h-dimensionale positiva, tali che:

$$\frac{\partial}{\partial t}\varphi(t, y, \eta) \geqq \varrho_0(\eta) \qquad \begin{cases} \eta \in I_\eta \\ (t, y) \in E(\eta) . \end{cases}$$

Poniamo, per $\eta \in I_\eta$:

$$\varrho(t, y, \eta) \begin{cases} = \varrho_0(\eta) & \text{se } (t, y) \in E(\eta) \\ = 0 & \text{se } (t, y) \in I_\xi - E(\eta) . \end{cases}$$

Per $\eta \in I_\eta$, $y \in I_y$, $t \in \bar{I}_t$, si ha allora:

$$\int_a^t \varrho(t', y, \eta)\, dt' \leqq \int_a^t \frac{\partial}{\partial t'} \varphi(t', y, \eta)\, dt' \leqq \varphi(t, y, \eta) .$$

Dalla (15) si trae quindi:

$$p_{\Phi_{\alpha(h)}}(I_\xi, \eta) = p(I_y, t, \eta) \geqq \int_{I_\xi} [p'_{\Phi_{\alpha(h)}}(t', y, \eta) + \varrho(t', y, \eta)]\, dy\, dt'$$

e ciò è assurdo.

Ripetendo per $q_{\Phi_{\alpha(h)}}$ le considerazioni ora svolte per $p_{\Phi_{\alpha(h)}}$ si perviene a dimostrare l'esistenza di una funzione $\psi(t, y, \eta)$ definita per $(y, \eta) \in I_y \times I_\eta$ e per $t \in \bar{I}_t$, la quale è non negativa, monotona non decrescente rispetto a t in $\bar{I}_t$, ha derivata rispetto a t quasi ovunque nulla, è misurabile e sommabile rispetto ad y in I_y ed è tale che:

$$(16) \qquad q(J_y, t, \eta) = \int_{J_y} \left[\psi(t, y, \eta) + \int_a^t q'_{\Phi_{\alpha(h)}}(t', y, \eta)\, dt'\right] dy \qquad \begin{cases} \eta \in I_\eta, \quad t \in \bar{I}_t \\ J_y \in \{J_y\} . \end{cases}$$

Per $\eta \in I_\eta$, $J_\xi \in \{J_\xi\}$ si ha:

$$\Phi_{\alpha(h)}(J_\xi, \eta) = p_{\Phi_{\alpha(h)}}(J_\xi, \eta) - q_{\Phi_{\alpha(h)}}(J_\xi, \eta) = p(J_y, t, \eta) - q(J_y, t, \eta) .$$

Dalle (14) e (16) segue allora, per $\eta \in I_\eta$, $t \in \bar{I}_t$, $J_y \in \{J_y\}$:

$$\begin{aligned} &\int_{J_y} [w(t, y, \eta) - w(a, y, \eta)]\, dy \\ &\qquad = \int_{J_y} \left\{\varphi(t, y, \eta) - \psi(t, y, \eta) + \int_a^t [p'_{\Phi_{\alpha(h)}}(t', y, \eta) - q'_{\Phi_{\alpha(h)}}(t', y, \eta)]\, dt'\right\} dy . \end{aligned}$$

Da questa relazione si trae l'esistenza di una funzione $\bar{u}$ definita in $\bar{I}$, equivalente a w (e quindi ad u) in I, tale che, per $t \in \bar{I}_t$, $(y, \eta) \in I_y \times I_\eta$, riesce:

$$\bar{u}(t, y, \eta) - w(a, y, \eta) = \varphi(t, y, \eta) - \psi(t, y, \eta) + \int_a^t [p'_{\Phi_{\alpha(h)}}(t', y, \eta) - q'_{\Phi_{\alpha(h)}}(t', y, \eta)]\, dt' .$$

Ne viene che $\bar{u}(t, y, \eta)$ è VL rispetto a t in I_t per ogni $(y, \eta) \in I_y \times I_\eta$ e che, quasi ovunque in I, riesce: $\frac{\partial \bar{u}}{\partial t} = \Phi'_{\alpha(h)}$, donde l'asserto, per $1 < h < r$. Nel caso $h = r$ la dimostrazione è analoga a quella ora esposta.

Bibliografia

[1] ADAMS, C. R., & J. A. CLARKSON: On definitions of bounded variation for functions of two variables. Trans. Amer. Math. Soc. **35**, 824—854 (1933).

[2] ADAMS, C. R., & J. A. CLARKSON: Properties of functions $f(x, y)$ of bounded variation. Trans. Amer. Math. Soc. **36**, 711—730 (1934).

[3] CALKIN, J. W.: Functions of real variables and absolute continuity. Duke Math. J. **6**, 170—186 (1940).

[4] CESARI, L.: Sulle funzioni a variazione limitata. Ann. R. Scuola Norm. Sup. Pisa Sc. Fis. Mat. II **5**, 299—313 (1936).

[5] DE VITO, L.: Sulle funzioni ad integrale di Dirichlet finito. Ann. Scuola Norm. Sup. Pisa, Sc. Fis. Mat. III **12**, 55—127 (1958).

[6] FICHERA, G.: Lezioni sulle trasformazioni lineari. Ist. Mat. Trieste **1955**.

[7] FICHERA, G.: Premesse ad una teoria generale dei problemi al contorno per le equazioni differenziali. Ist. Alta Matematica, Roma **1958**.

[8] FLEMING, W. H.: Functions with generalized gradient and generalized surfaces. Ann. Mat. Pura e Appl. IV **44**, 93—103 (1957).

[9] GOFFMAN, C.: Lower semicontinuity and area functionals. Rend. Circ. Mat. Palermo II **2**, 203—234 (1953).

[10] KRICKEBERG, K.: Distributionen, Funktionen beschränkter Variation und Lebesguescher Inhalt nichtparametrischer Flächen. Ann. Mat. Pura ed Appl. IV **44**, 105—133 (1957).

[11] MORREY, C. B.: Functions of several variables and absolute continuity. Duke Math. J. **6**, 187—215 (1940).

[12] PAUC, CHR.: Considerations sur les gradients généralisés de G. FICHERA et E. DE GIORGI. Ann. Mat. Pura ed Appl. IV **40**, 307—312 (1955).

[13] PAUC, CHR.: Functions with Generalized Gradients in the Theory of Cell Functions. Ann. Mat. Pura ed Appl. IV **44**, 135—152 (1957).

[14] PICONE, M., e T. VIOLA: Lezioni sulla Teoria Moderna dell'integrazione. Edizioni Scientifiche Einaudi. 1952.

[15] SAKS, S.: Theory of the Integral, II. ed. New York: Hafner Publ. Co. 1937.

[16] STAMPACCHIA, G.: Gli integrali doppi del calcolo delle variazioni in forma ordinaria. Rend. Accad. Naz. Lincei VIII **8** (1950).

[17] TONELLI, L.: Sulla quadratura della superficie. Atti R. Accad. naz. Lincei VI **3**, 357—362, 445—450, 633—638 (1926).

[18] TONELLI, L.: Sulle funzioni di due variabili assolutamente continue. Mem. R. Accad. Sc. Ist. Bologna, Sc. fis. VIII **6**, 81—88 (1928/29).

[19] TONELLI, L.: Sulle funzioni di due variabili generalmente a variazione limitata. Ann. R. Scuola Norm. Sup. Pisa Sc. fis. mat. II **5**, 299—313 (1936).

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The Mechanics of Non-Linear Materials with Memory

Part II

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1. Introduction

In a previous paper (Green & Rivlin 1957), subsequently referred to as Part I, the form of the constitutive equations governing the deformation of a class of materials possessing memory was discussed. It was assumed that the stress in an element of the material depends not only on the deformation gradients in the element at the instant of time considered, but also on those at previous instants of time. The limitations imposed on the constitutive equation by the fact that it must be form-invariant under a rotation of the physical system considered (consisting of the body and applied forces) were examined.

This was done by first considering that the stress depends on the deformation gradients at a number of discrete times up to the instant of measurement. Then, the number of instants of time was considered to increase indefinitely, so that the expression for the stress became a functional of the deformation gradients. In this analysis it was found that the form-invariance of the constitutive equation under a rotation of the physical system leads naturally to a particular form of dependence of the stress on the deformation gradients at the instant of measurement. It was assumed that, apart from this, the expression for the stress as a functional of the deformation gradients at times up to and including the instant of measurement is continuous.

In the present paper, we do not make this assumption, but allow that the stress may have arbitrary polynomial dependence on the deformation gradients at the instant of measurement, while its functional dependence on the deformation gradients at times preceding the instant of measurement is continuous. Under these conditions, the limitations imposed by isotropy of the material in its undeformed state on the form of the constitutive equation is considered.

2. Special Dependence of the Stress on the Displacement Gradients

We consider a three-dimensional body to undergo deformation described in a fixed rectangular Cartesian coordinate system x by

$$\begin{aligned} x_i(\tau) &= x_i(X_j, \tau) \qquad (\tau > 0), \\ x_i(\tau) &= X_i \qquad\qquad\;\; (\tau \leqq 0), \end{aligned} \tag{2.1}$$

where X_i and x_i are the coordinates in the system x of a generic particle of the body at zero time and time τ respectively. We assume that $x_i(\tau)$ are single-

valued functions of the arguments, possessing continuous spatial derivatives up to any required order except possibly at singular points, lines and surfaces. If the deformation is to be possible in a real material we must have

$$|\partial x_i(\tau)/\partial X_j| > 0.$$

We assume the stress components $\sigma_{ij}(t)$ at time t in the system x to be polynomial functions of the deformation gradients $\partial x_i(\tau_\alpha)/\partial X_j$ $(\alpha = 0, 1, 2, \ldots, N)$ at the $N+1$ instants of time $\tau_1, \tau_2, \ldots, \tau_N, \tau_0(=t)$ between $\tau = 0$ and $\tau = t$. It has been shown (GREEN & RIVLIN 1957), that σ_{ij} may then be expressed in the form

$$\sigma_{ij} = \frac{1}{\sqrt{g}}\left[f\,\delta_{ij} + \frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s} f_{rs}\right], \tag{2.2}$$

where the notation

$$\begin{gathered} \sigma_{ij} = \sigma_{ij}(t), \qquad \frac{\partial x_i}{\partial X_r} = \frac{\partial x_i(t)}{\partial X_r}, \\ g_{pq}(\tau) = \frac{\partial x_k(\tau)}{\partial X_p}\frac{\partial x_k(\tau)}{\partial X_q}, \qquad g_{pq} = g_{pq}(t), \\ g(\tau) = |g_{pq}(\tau)|, \qquad g = g(t), \end{gathered} \tag{2.3}$$

is used, δ_{ij} denotes the Kronecker delta and f and f_{rs} are polynomial functions of $g_{pq}(\tau_\alpha)$ and $\sqrt{g(\tau_\alpha)}$ $(\alpha = 0, 1, 2, \ldots, N)$.

We may re-write (2.2) in a somewhat more succinct form by observing that

$$2g(t)\,\delta_{ij} = \frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s} e_{rmn}\, e_{suv}\, g_{mu}\, g_{nv}, \tag{2.4}$$

so that

$$\sigma_{ij} = \frac{1}{2g^{\frac{3}{2}}}\frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s} F_{rs}, \tag{2.5}$$

where

$$F_{rs} = f\, e_{rmn}\, e_{suv}\, g_{mu}\, g_{nv} + 2g\, f_{rs}. \tag{2.6}$$

Hence F_{rs} is a symmetric polynomial function of $g_{pq}(\tau_\alpha)$ and $\sqrt{g(\tau_\alpha)}$ $(\alpha = 0, 1, 2, \ldots, N)$.

3. Isotropic Materials

In Part I the restrictions required for a material which is isotropic when $\tau \leqq 0$ were obtained using (2.2) and the corresponding results for (2.5) can then be deduced*. Here, however, we proceed directly from (2.5). Let $\bar{x}$ be a fixed rectangular Cartesian coordinate system related to x by

$$\bar{x}_i = A_{ij}\, x_j, \tag{3.1}$$

where A_{ij} are constants satisfying the orthogonality conditions

$$A_{ir}A_{jr} = A_{ri}A_{rj} = \delta_{ij}, \quad |A_{ij}| = 1. \tag{3.2}$$

* A slight gap in the argument in Part I was completed later (GREEN & RIVLIN 1958).

If $\bar{x}_i(\tau)$ denotes the coordinates of $x_i(\tau)$ in the system $\bar{x}$ and $\bar{X}_i = \bar{x}_i(0)$, we have, from (3.1),

$$\bar{x}_i(\tau) = A_{ij}\,x_j(\tau), \qquad \bar{X}_i = A_{ij}\,X_j,$$
$$\bar{g}_{ij}(\tau) = \frac{\partial \bar{x}_r(\tau)}{\partial \bar{X}_i}\,\frac{\partial \bar{x}_r(\tau)}{\partial \bar{X}_j} = A_{ir}\,A_{js}\,g_{rs}(\tau), \tag{3.3}$$
$$\bar{g}(\tau) = |\bar{g}_{ij}(\tau)| = g(\tau).$$

Denoting the stress in the system $\bar{x}$ by $\bar{\sigma}_{ij}$ we obtain

$$\bar{\sigma}_{ij} = A_{im}\,A_{jn}\,\sigma_{mn}. \tag{3.4}$$

Hence, from (2.5), (3.1) and (3.4),

$$\bar{\sigma}_{ij} = \frac{1}{2g^{\frac{3}{2}}}\,\frac{\partial \bar{x}_i}{\partial \bar{X}_r}\,\frac{\partial \bar{x}_j}{\partial \bar{X}_s}\,A_{rm}\,A_{sn}\,F_{mn}. \tag{3.5}$$

If the material is isotropic at $\tau = 0$ then

$$\bar{\sigma}_{ij} = \frac{1}{2\bar{g}^{\frac{3}{2}}}\,\frac{\partial \bar{x}_i}{\partial \bar{X}_r}\,\frac{\partial \bar{x}_j}{\partial \bar{X}_s}\,\left[F_{rs}\,\bar{g}_{pq}(\tau_\alpha),\,\sqrt{\bar{g}(\tau_\alpha)}\right], \tag{3.6}$$

so that, using (3.5) and (3.6), we have

$$\frac{\partial \bar{x}_i}{\partial \bar{X}_r}\,\frac{\partial \bar{x}_j}{\partial \bar{X}_s}\left\{F_{rs}\left[\bar{g}_{pq}(\tau_\alpha),\,\sqrt{\bar{g}(\tau_\alpha)}\right] - A_{rm}\,A_{sn}\,F_{mn}\left[g_{pq}(\tau_\alpha),\,\sqrt{g(\tau_\alpha)}\right]\right\} = 0. \tag{3.7}$$

Multiplying this equation by the non-zero expressions

$$\frac{\partial \bar{X}_k}{\partial \bar{x}_i}\,\frac{\partial \bar{X}_l}{\partial \bar{x}_j},$$

we obtain

$$F_{rs}\left[\bar{g}_{pq}(\tau_\alpha),\,\sqrt{g(\tau_\alpha)}\right] = A_{rm}\,A_{sn}\,F_{mn}\left[g_{pq}(\tau_\alpha),\,\sqrt{g(\tau_\alpha)}\right]. \tag{3.8}$$

Using the notation

$$\boldsymbol{g}(\tau) = \|g_{ij}(\tau)\|, \qquad \bar{\boldsymbol{g}}(\tau) = \|\bar{g}_{ij}(\tau)\|, \tag{3.9}$$

it follows that F_{rs} are the components of a symmetric matrix polynomial in the matrices $\boldsymbol{g}(\tau_\alpha)$ in which the coefficients are scalar polynomials in $\sqrt{g(\tau_\alpha)}$ ($\alpha = 0, 1, 2, \ldots, N$) and traces of products formed from the matrices $\boldsymbol{g}(\tau_\alpha)$ ($\alpha = 0, 1, 2, \ldots, N$). Since $[g(\tau)]^{\frac{1}{2}}$ is a continuous single-valued function of $g_{pq}(\tau)$ and $1/g^{\frac{3}{2}}$ has no singularities, we can omit the factor $1/(2g^{\frac{3}{2}})$ and the arguments $[g(\tau_\alpha)]^{\frac{1}{2}}$ in (2.5) and write, to any desired degree of approximation,

$$\boldsymbol{\sigma} = \boldsymbol{c}\,\boldsymbol{F}\,\boldsymbol{c}', \tag{3.10}$$

where

$$\boldsymbol{\sigma} = \|\sigma_{ij}\|, \qquad \boldsymbol{c} = \|\partial x_i/\partial X_r\|. \tag{3.11}$$

In (3.10) $\boldsymbol{c}'$ is the transpose of $\boldsymbol{c}$ and $\boldsymbol{F}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{g}(\tau_\alpha)$ in which the coefficients are scalar polynomial functions in the traces of products formed from the matrices $\boldsymbol{g}(\tau_\alpha)$ ($\alpha = 0, 1, 2, \ldots, N$).

The expression (3.10), valid for materials which are isotropic at time $\tau \leqq 0$, can be written in a convenient alternative form. Let*

$$\begin{aligned} h_{pq}(\tau) &= \frac{\partial X_r}{\partial x_p} \frac{\partial X_s}{\partial x_q} g_{rs}(\tau) = \frac{\partial x_m(\tau)}{\partial x_p} \frac{\partial x_m(\tau)}{\partial x_q}, \\ \boldsymbol{h}(\tau) &= \|h_{pq}(\tau)\|, \end{aligned} \tag{3.12}$$

and

$$\boldsymbol{C} = \|C_{ij}\|, \qquad C_{ij} = \frac{\partial x_i}{\partial X_m} \frac{\partial x_j}{\partial X_m}. \tag{3.13}$$

Then

$$\boldsymbol{C} = \boldsymbol{c}\,\boldsymbol{c}', \quad \boldsymbol{g}(\tau) = \boldsymbol{c}'\,\boldsymbol{h}(\tau)\,\boldsymbol{c}, \quad \boldsymbol{h}(\tau) = (\boldsymbol{c}')^{-1}\boldsymbol{g}(\tau)\,\boldsymbol{c}^{-1}. \tag{3.14}$$

The trace of products of matrices formed from $\boldsymbol{g}(\tau_\alpha)$ $(\alpha = 0, 1, 2, \ldots, N)$ can be expressed as the trace of products of matrices formed from $\boldsymbol{C}$ and $\boldsymbol{h}(\tau_\alpha)$ $(\alpha = 0, 1, 2, \ldots, N)$. Also a symmetric matrix polynomial formed from the matrices $\boldsymbol{g}(\tau_\alpha)$ can be expressed in the form $\boldsymbol{c}'\boldsymbol{L}\boldsymbol{c}$ where $\boldsymbol{L}$ is a symmetric matrix polynomial formed from the matrices $\boldsymbol{C}$ and $\boldsymbol{h}(\tau_\alpha)$. It follows that we may write (3.10) in the alternative form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{C}, \boldsymbol{h}(\tau_\alpha)], \tag{3.15}$$

where $\boldsymbol{\sigma}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{C}, \boldsymbol{h}(\tau_\alpha)$ $(\alpha = 1, 2, \ldots, N)$, with coefficients which are polynomial functions of the traces of products formed from the matrices $\boldsymbol{C}, \boldsymbol{h}(\tau_\alpha)$ $(\alpha = 1, 2, \ldots, N)$.

Conversely, we can show that any constitutive equation of the form (3.15) can be expressed with any desired accuracy in the form (3.10), where $\boldsymbol{F}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{g}(\tau_\alpha)$ with coefficients which are scalar polynomial functions in the traces of products formed from these matrices.

4. Isotropic Tensor Functional

When the material is isotropic initially we may make the transition from tensor functions to tensor functionals using either (3.10) or (3.15) as a starting point. In order to keep in line with the work of Part I we use (3.10) and suppose that

$$\sigma_{ij} = \frac{\partial x_i}{\partial X_r} \frac{\partial x_j}{\partial X_s} F_{rs}, \tag{4.1}$$

where we assume that F_{rs} is a symmetric matrix polynomial in g_{pq} with coefficients which are functionals of $g_{pq}(\tau)$ in the range $0 \leqq \tau \leqq t$. We assume that these functionals are continuous functionals of $g_{pq}(\tau)$ over the compact aggregate of functions which are continuous in the range $0 \leqq \tau \leqq t$. We denote the Fourier half-range cosine coefficients of $g_{pq}(\tau)$ by $G_{pq}^{(\alpha)}$, where

$$\begin{aligned} G_{pq}^{(\alpha)} &= \frac{2}{t} \int_0^t g_{pq}(\tau) \cos \frac{\alpha \pi \tau}{t} \, d\tau \qquad (\alpha > 0), \\ G_{pq}^{(0)} &= \frac{1}{t} \int_0^t g_{pq}(\tau) \, d\tau. \end{aligned} \tag{4.2}$$

* From (2.1) $x_m(\tau)$ may be expressed as a function of $x_m(t)$, τ, t by eliminating X_j, and then $\partial x_m(\tau)/\partial x_p$ can be evaluated. Alternatively, we can use the first expression in (3.12) to find $h_{pq}(\tau)$.

Then each continuous functional in the expression for F_{rs} may be approximated by a polynomial $P^{(\alpha)}(G_{pq}^{(0)}, G_{pq}^{(1)}, \ldots, G_{pq}^{(\alpha)})$ which tends uniformly to the functional as $\alpha \to \infty$. It follows that F_{rs} can be expressed with any desired approximation by a polynomial in g_{pq} and $G_{pq}^{(0)}, G_{pq}^{(1)}, \ldots, G_{pq}^{(N)}$ with a sufficiently large choice of N and we write

$$\sigma_{ij} = \frac{\partial x_i}{\partial X_r} \frac{\partial x_j}{\partial X_s} F_{rs}[g_{pq}, G_{pq}^{(0)}, G_{pq}^{(1)}, \ldots, G_{pq}^{(N)}], \tag{4.3}$$

where F_{rs} is a polynomial.

We next consider a change of rectangular Cartesian axes of the form (3.1) and (3.2). If $\bar{\sigma}_{ij}$ denotes the stress components at time t in the system $\bar{x}$, then $\bar{\sigma}_{ij}$ is given by (3.4) and (4.3). Also, if the material is isotropic for $\tau \leqq 0$, then

$$\bar{\sigma}_{ij} = \frac{\partial \bar{x}_i}{\partial \bar{X}_r} \frac{\partial \bar{x}_j}{\partial \bar{X}_s} F_{rs}[\bar{g}_{pq}, \bar{G}_{pq}^{(0)}, \bar{G}_{pq}^{(1)}, \ldots, \bar{G}_{pq}^{(N)}], \tag{4.4}$$

where

$$\left.\begin{aligned} \bar{G}_{pq}^{(\alpha)} &= \frac{2}{t} \int_0^t \bar{g}_{pq}(\tau) \cos \frac{\alpha \pi \tau}{t} \, d\tau \qquad (\alpha > 0), \\ \bar{G}_{pq}^{(0)} &= \frac{1}{t} \int_0^t \bar{g}_{pq}(\tau) \, d\tau, \end{aligned}\right\} \tag{4.5}$$

and $\bar{g}_{pq}(\tau)$ is given by (3.3). From (3.3), (4.2) and (4.5), we have

$$\left.\begin{aligned} \bar{g}_{pq} &= A_{pm} A_{qn} g_{mn}, \\ \bar{G}_{pq}^{(\alpha)} &= A_{pm} A_{qn} G_{mn}^{(\alpha)} \qquad (\alpha \geqq 0). \end{aligned}\right\} \tag{4.6}$$

Also, from (3.1), (3.3), (3.4), (4.3) and (4.4), we obtain

$$\frac{\partial \bar{x}_i}{\partial \bar{X}_r} \frac{\partial \bar{x}_j}{\partial \bar{X}_s} \{F_{rs}[\bar{g}_{pq}, \bar{G}_{pq}^{(\alpha)}] - A_{rm} A_{sn} F_{mn}[g_{pq}, G_{pq}^{(\alpha)}]\} = 0$$

and hence

$$F_{rs}[\bar{g}_{pq}, \bar{G}_{pq}^{(\alpha)}] = A_{rm} A_{sn} F_{mn}[g_{pq}, G_{pq}^{(\alpha)}]. \tag{4.7}$$

Using the notation

$$\boldsymbol{G}_\alpha = \|G_{pq}^{(\alpha)}\|, \tag{4.8}$$

it follows from (4.7) and (4.6) that $\boldsymbol{F}$ is a symmetric matrix polynomial in the symmetric matrices $\boldsymbol{g}$, $\boldsymbol{G}_\alpha$ $(\alpha = 0, 1, \ldots, N)$ and equation (4.3) may be written in the matrix form

$$\boldsymbol{\sigma} = \boldsymbol{c} \boldsymbol{F} \boldsymbol{c}', \tag{4.9}$$

where $\boldsymbol{F}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{g}$, $\boldsymbol{G}_\alpha$ $(\alpha = 0, 1, \ldots, N)$.

Since $\boldsymbol{g}(\tau) = \boldsymbol{c}' \boldsymbol{h}(\tau) \boldsymbol{c}$, it follows that $\boldsymbol{G}_\alpha = \boldsymbol{c}' \boldsymbol{H}_\alpha \boldsymbol{c}$, where

$$\left.\begin{aligned} \boldsymbol{H}_\alpha &= \|H_{pq}^{(\alpha)}\|, \\ H_{pq}^{(\alpha)} &= \frac{2}{t} \int_0^t h_{pq}(\tau) \cos \frac{\alpha \pi \tau}{t} \, d\tau \qquad (\alpha > 0), \\ H_{pq}^{(0)} &= \frac{1}{t} \int_0^t h_{pq}(\tau) \, d\tau. \end{aligned}\right\} \tag{4.10}$$

Hence, by an argument similar to that used in § 3, we may reduce (4.9) to the alternative form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{C}, \boldsymbol{H}_0, \boldsymbol{H}_1, \ldots, \boldsymbol{H}_N], \tag{4.11}$$

where $\boldsymbol{\sigma}$ is a symmetric matrix polynomial in the arguments stated and $\boldsymbol{C}$ is given by (3.13).

5. Further Development of the Equations for Isotropic Materials

Since $\boldsymbol{F}$ in (4.9) is a symmetric matrix polynomial in the matrices $\boldsymbol{g}$, $\boldsymbol{G}_\alpha$ ($\alpha = 0, 1, \ldots, N$) it is apparent that it may be expressed as the sum of a number of terms of the form

$$\begin{aligned} f(t) \int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_1 \pi \tau_1}{t} \cos \frac{\alpha_2 \pi \tau_2}{t} \cdots \cos \frac{\alpha_P \pi \tau_P}{t} \times \\ \times (\operatorname{tr} \boldsymbol{\Pi}_1 \operatorname{tr} \boldsymbol{\Pi}_2 \ldots \operatorname{tr} \boldsymbol{\Pi}_\mu) (\boldsymbol{\Pi} + \boldsymbol{\Pi}')\, d\tau_1\, d\tau_2 \ldots d\tau_P, \end{aligned} \tag{5.1}$$

where $\boldsymbol{\Pi}_1, \boldsymbol{\Pi}_2, \ldots, \boldsymbol{\Pi}_\mu$ and $\boldsymbol{\Pi}$ are matrix products formed from $\boldsymbol{g}, \boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_P)$ and are such that $\boldsymbol{\Pi}_1 \boldsymbol{\Pi}_2 \ldots \boldsymbol{\Pi}_\mu \boldsymbol{\Pi}$ is linear in each of the matrices $\boldsymbol{g}(\tau_1)$, $\boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_P)$, and $\boldsymbol{\Pi}'$ denotes the transpose of $\boldsymbol{\Pi}$. Let $\boldsymbol{\Pi}_1$ be a matrix product formed from $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_{\beta_1})$ and possibly $\boldsymbol{g}$; let $\boldsymbol{\Pi}_2$ be a matrix product formed from $\boldsymbol{g}(\tau_{\beta_1+1}), \boldsymbol{g}(\tau_{\beta_1+2}), \ldots, \boldsymbol{g}(\tau_{\beta_2})$ and possibly $\boldsymbol{g}$ and so on, so that $\boldsymbol{\Pi}_\mu$ is a matrix product formed from $\boldsymbol{g}(\tau_{\beta_{\mu-1}+1}), \boldsymbol{g}(\tau_{\beta_{\mu-1}+2}), \ldots, \boldsymbol{g}(\tau_{\beta_\mu})$ and possibly $\boldsymbol{g}$, and let $\boldsymbol{\Pi}$ be a matrix product formed from $\boldsymbol{g}(\tau_{\beta_\mu+1}), \boldsymbol{g}(\tau_{\beta_\mu+2}), \ldots, \boldsymbol{g}(\tau_P)$ and possibly $\boldsymbol{g}$. Then, the term (5.1) may be re-written as

$$\begin{aligned} & f(t) \int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_1 \pi \tau_1}{t} \cos \frac{\alpha_2 \pi \tau_2}{t} \cdots \cos \frac{\alpha_{\beta_1} \pi \tau_{\beta_1}}{t} \operatorname{tr} \boldsymbol{\Pi}_1\, d\tau_1 \ldots d\tau_{\beta_1} \\ & \int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_{\beta_1+1} \pi \tau_{\beta_1+1}}{t} \cdots \cos \frac{\alpha_{\beta_2} \pi \tau_{\beta_2}}{t} \operatorname{tr} \boldsymbol{\Pi}_2\, d\tau_{\beta_1+1} \ldots d\tau_{\beta_2} \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_{\beta_{\mu-1}+1} \pi \tau_{\beta_{\mu-1}+1}}{t} \cdots \cos \frac{\alpha_{\beta_\mu} \pi \tau_{\beta_\mu}}{t} \operatorname{tr} \boldsymbol{\Pi}_\mu\, d\tau_{\beta_{\mu-1}+1} \ldots d\tau_{\beta_\mu} \\ & \int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_{\beta_\mu+1} \pi \tau_{\beta_\mu+1}}{t} \cdots \cos \frac{\alpha_P \pi \tau_P}{t} (\boldsymbol{\Pi} + \boldsymbol{\Pi}')\, d\tau_{\beta_\mu+1} \ldots d\tau_P. \end{aligned} \tag{5.2}$$

We now consider the factor

$$\int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_1 \pi \tau_1}{t} \cos \frac{\alpha_2 \pi \tau_2}{t} \cdots \cos \frac{\alpha_{\beta_1} \pi \tau_{\beta_1}}{t} \operatorname{tr} \boldsymbol{\Pi}_1\, d\tau_1 \ldots d\tau_{\beta_1}. \tag{5.3}$$

It has been shown (SPENCER & RIVLIN 1959) that the trace of a matrix product formed from the R symmetric 3×3 matrices $\boldsymbol{a}_P$ ($P = 1, 2, \ldots, R$) may be expressed as a polynomial in the traces of matrix products formed from those listed in Table 1 by replacing $K_1, K_2, \ldots, K_7$ by all permutations of $1, 2, \ldots, R$ seven at a time.

Since $\boldsymbol{\Pi}_1$ is linear in each of the matrices $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_{\beta_1})$, $\operatorname{tr}\boldsymbol{\Pi}_1$ must be expressible as a polynomial in the invariants listed in Table 2 and the invariants formed from these by replacing $\tau_1, \tau_2, \ldots, \tau_7$ by all possible permutations of $\tau_1, \tau_2, \ldots, \tau_{\beta_1}$ seven at a time.

Table 1

$$
\begin{array}{lll}
\operatorname{tr} \boldsymbol{a}_{K_1}, & \operatorname{tr} \boldsymbol{a}_{K_1}^2, & \operatorname{tr} \boldsymbol{a}_{K_1}^3; \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2}, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2}^2, & \operatorname{tr} \boldsymbol{a}_{K_1}^2 \boldsymbol{a}_{K_2}^2; \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3}, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3}^2, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2}^2 \boldsymbol{a}_{K_3}^2; \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}^2, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3}^2 \boldsymbol{a}_{K_4}^2, \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}^2; & &
\end{array}
$$

$$
\begin{array}{ll}
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2, \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2, \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}^2 \boldsymbol{a}_{K_5}^2; \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6}, & \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6}^2; \\
\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6} \boldsymbol{a}_{K_7}. &
\end{array}
$$

Denoting the invariants in Table 2 which involve none of the matrices $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_7)$ by $\operatorname{tr}\tilde{\boldsymbol{\omega}}_\alpha^{(0)}$ $(\alpha = 1, 2, 3)$, those which involve $\boldsymbol{g}(\tau_1)$ only or $\boldsymbol{g}$ and $\boldsymbol{g}(\tau_1)$ only by $\operatorname{tr}\tilde{\boldsymbol{\omega}}_\alpha^{(1)}$ $(\alpha = 1, 2, 3)$, those which involve $\boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$

Table 2

$$
\begin{array}{l}
\operatorname{tr}\boldsymbol{g}, \quad \operatorname{tr}\boldsymbol{g}^2, \quad \operatorname{tr}\boldsymbol{g}^3; \\
\operatorname{tr}\boldsymbol{g}\,\boldsymbol{g}(\tau_1), \quad \operatorname{tr}\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1); \\
\operatorname{tr}\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2), \quad \operatorname{tr}\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2); \\
\operatorname{tr}\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3), \quad \operatorname{tr}\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3), \\
\operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}^2; \\
\operatorname{tr}\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4), \quad \operatorname{tr}\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4), \\
\operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}^2, \quad \operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}^2, \\
\operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}^2; \\
\operatorname{tr}\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5), \quad \operatorname{tr}\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5); \\
\operatorname{tr}\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)\,\boldsymbol{g}(\tau_6); \\
\operatorname{tr}\boldsymbol{g}(\tau_1), \quad \operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2), \quad \operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3), \\
\operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4), \quad \operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5), \\
\operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)\,\boldsymbol{g}(\tau_6), \\
\operatorname{tr}\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)\,\boldsymbol{g}(\tau_6)\,\boldsymbol{g}(\tau_7).
\end{array}
$$

or $\boldsymbol{g}$, $\boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$ only by $\operatorname{tr}\tilde{\boldsymbol{\omega}}_\alpha^{(2)}$ $(\alpha = 1, 2, 3)$, and so on, we see that the term (5.3) may be expressed as a polynomial in expressions of the forms

$$
\begin{gathered}
\operatorname{tr}\tilde{\boldsymbol{\omega}}_\alpha^{(0)}, \quad \int_0^t \varphi(\tau_1) \operatorname{tr}\tilde{\boldsymbol{\omega}}_\alpha^{(1)}\, d\tau_1, \quad \int_0^t\!\int_0^t \varphi(\tau_1, \tau_2) \operatorname{tr}\tilde{\boldsymbol{\omega}}_\alpha^{(2)}\, d\tau_1\, d\tau_2, \\
\cdots \int_0^t \cdots \int_0^t \varphi(\tau_1, \tau_2, \ldots, \tau_7) \operatorname{tr}\tilde{\boldsymbol{\omega}}_\alpha^{(7)}\, d\tau_1\, d\tau_2 \ldots d\tau_7,
\end{gathered} \tag{5.4}
$$

where the functions $\varphi(\tau_1), \varphi(\tau_1, \tau_2), \ldots, \varphi(\tau_1, \tau_2, \ldots, \tau_7)$ are analytic functions of their arguments. Each of the other factors in (5.2) except the last may be similarly expressed.

We now consider the last factor in (5.2), *i.e.*

$$\int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_{\beta\mu+1}\,\pi\,\tau_{\beta\mu+1}}{t} \cdots \cos \frac{\alpha_P\,\pi\,\tau_P}{t} (\boldsymbol{\Pi} + \boldsymbol{\Pi}')\, d\tau_{\beta\mu+1} \dots d\tau_P. \tag{5.5}$$

Since $\boldsymbol{\Pi} + \boldsymbol{\Pi}'$ is a symmetric isotropic matrix polynomial in the matrices $\boldsymbol{g}(\tau_{\beta\mu+1}), \boldsymbol{g}(\tau_{\beta\mu+2}), \dots, \boldsymbol{g}(\tau_P)$ and $\boldsymbol{g}$, linear in each of them except $\boldsymbol{g}$, it follows (Spencer & Rivlin 1958) that it may be expressed as an isotropic matrix polynomial of the form

$$\boldsymbol{\Pi} + \boldsymbol{\Pi}' = \sum \varphi_\nu (\boldsymbol{\chi}_\nu + \boldsymbol{\chi}_\nu'), \tag{5.6}$$

where $\boldsymbol{\chi}_\nu$ denote the matrix products formed from those listed in Table 3 by replacing $\tau_1, \tau_2, \dots, \tau_6$ by all possible permutations of $\tau_{\beta\mu+1}, \tau_{\beta\mu+2}, \dots, \tau_P$ six at a time, while φ_ν are polynomials in the invariants obtained from those listed in Table 2, by replacing $\tau_1, \tau_2, \dots, \tau_7$ by all possible permutations of $\tau_{\beta\mu+1}, \tau_{\beta\mu+2}, \dots, \tau_P$ seven at a time. $\boldsymbol{\chi}'$ denotes the transpose of $\boldsymbol{\chi}$. Since $\boldsymbol{\Pi} + \boldsymbol{\Pi}'$ is linear in each of the matrices $\boldsymbol{g}(\tau_{\beta\mu+1}), \boldsymbol{g}(\tau_{\beta\mu+2}), \dots, \boldsymbol{g}(\tau_P)$, φ_ν and $\boldsymbol{\chi}_\nu + \boldsymbol{\chi}_\nu'$ cannot involve any of these matrices in common. We may therefore express the factor (5.5) as an isotropic matrix polynomial in which the matrix terms are of the forms

$$\begin{aligned} &\boldsymbol{\chi}_\nu^{(0)}, \int_0^t \psi_\nu(\tau_1)\,(\boldsymbol{\chi}_\nu^{(1)} + \boldsymbol{\chi}_\nu^{(1)\prime})\, d\tau_1, \\ &\int_0^t \int_0^t \psi_\nu(\tau_1, \tau_2)\,(\boldsymbol{\chi}_\nu^{(2)} + \boldsymbol{\chi}_\nu^{(2)\prime})\, d\tau_1\, d\tau_2, \dots, \\ &\int_0^t \int_0^t \cdots \int_0^t \psi_\nu(\tau_1, \tau_2, \dots, \tau_6)\,(\boldsymbol{\chi}_\nu^{(6)} + \boldsymbol{\chi}_\nu^{(6)\prime})\, d\tau_1\, d\tau_2 \dots d\tau_6, \end{aligned} \tag{5.7}$$

where $\boldsymbol{\chi}_\nu^{(0)}$ are the matrix products listed in Table 3 which do not involve any of the matrices $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \dots, \boldsymbol{g}(\tau_6)$, $\boldsymbol{\chi}_\nu^{(1)}$ are those which involve only $\boldsymbol{g}(\tau_1)$

Table 3

$\boldsymbol{I}$;

$\boldsymbol{g}$, $\boldsymbol{g}^2$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)$, $\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)$,
$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}^2$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$,
$\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$,
$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}^2$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}^2$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}^2$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_3)$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$, $\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$;

$\boldsymbol{g}(\tau_1)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)\,\boldsymbol{g}(\tau_6)$.

or $\boldsymbol{g}$ and $\boldsymbol{g}(\tau_1)$, $\boldsymbol{\chi}_\nu^{(2)}$ are those which involve only $\boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$ or $\boldsymbol{g}$, $\boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$ and so on, and the functions ψ are analytic functions of their arguments.

It follows that (5.2) and hence $\boldsymbol{F}$ may be expressed as an isotropic matrix polynomial in which the matrix terms take the forms (5.7) and the coefficients are polynomials in the invariants (5.4) and functions of t. In general, the matrix polynomial will contain more than one term of each of the forms (5.7), except the first. Alternatively, we can bring the coefficients under the integration signs in the terms of the forms (5.7) to derive the result that $\boldsymbol{F}$ may be expressed in the form

$$\begin{aligned}\boldsymbol{F} = & \sum_{\nu} \vartheta_{\nu} \boldsymbol{\chi}_{\nu}^{(0)} + \sum_{\nu} \int_0^t \vartheta_{\nu}(\tau_1)\, (\boldsymbol{\chi}_{\nu}^{(1)} + \boldsymbol{\chi}_{\nu}^{(1)\prime})\, d\tau_1 + \\ & + \sum_{\nu} \int_0^t \int_0^t \vartheta_{\nu}(\tau_1, \tau_2)\, (\boldsymbol{\chi}_{\nu}^{(2)} + \boldsymbol{\chi}_{\nu}^{(2)\prime})\, d\tau_1\, d\tau_2 + \cdots + \\ & + \sum_{\nu} \int_0^t \int_0^t \cdots \int_0^t \vartheta_{\nu}(\tau_1, \tau_2, \ldots, \tau_6)\, (\boldsymbol{\chi}_{\nu}^{(6)} + \boldsymbol{\chi}_{\nu}^{(6)\prime})\, d\tau_1\, d\tau_2 \ldots d\tau_6,\end{aligned} \tag{5.8}$$

where the ϑ's are functions of t and of their indicated arguments and polynomials in the invariants (5.4).

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References

Green, A. E., & R. S. Rivlin: Arch. Rational Mech. Anal. **1**, 1 (1957).
Green, A. E., & R. S. Rivlin: Arch. Rational Mech. Anal. **1**, 470 (1958).
Spencer, A. J. M., & R. S. Rivlin: Arch. Rational Mech. Anal. **2**, 309 (1959).

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Mathematische Begründung der Scheibentheorie (zweidimensionale Elastizitätstheorie)

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I. Problemstellung

In der Begründung der Elastizitätstheorie gewinnt man aus dem Hookeschen Gesetz unter gewissen, bekannten Vernachlässigungen die Fundamentalgleichungen der linearen dreidimensionalen Elastizitätstheorie: Für das Spannungsfeld σ_{ik} und das Verschiebungsfeld u_i gilt mit dem daraus abgeleiteten Verzerrungsfeld[1]

(1) $$\gamma_{ik} = \tfrac{1}{2}(u_{i|k} + u_{k|i}),$$

(2a) $$\gamma_{ik} = \frac{1}{2G}\sigma_{ik} - \delta_{ik}\frac{\sum \sigma_{jj}}{Em}$$

bzw. die Umkehrung

(2b) $$\sigma_{ik} = 2G\gamma_{ik} + \frac{2G}{m-2}\delta_{ik}(\sum \gamma_{ll})$$

und

(3) $$\sigma_{ik|k} = 0.$$

Elimination ergibt dann

(4) $$G\left\{u_{i|kk} + \frac{m}{m-2}u_{k|ik}\right\} = 0.$$

Dabei gilt zwischen den drei positiven Konstanten E (Elastizitätsmodul) G (Gleit- oder Schubmodul) und m (Querkontraktionszahl $\geqq 2$) die Relation $2G(m+1) = Em$. Zur Bestimmung des Spannungs- und Verschiebungszustandes gehören zu den Differentialgleichungen (1) bis (4), die Innern eines Bereiches gelten, noch Randbedingungen:

(5)
a) $n_i\sigma_{ik} = f_k$ an den Teilen der Oberfläche (mit der Normalen n_i), wo die Oberflächenkraftdichte f_k gegeben ist.
b) vorgeschriebene Werte für u_i, wo die Verschiebung gegeben ist.

---

[1] Indizes bei u, σ, γ durchlaufen 1, 2, 3; bei den späteren Buchstaben v, τ, ζ durchlaufen die Indizes 1, 2. Summationskonvention und Ableitungsbezeichnung $w_{|i} = \frac{\partial}{\partial x_i} w$ wird verwendet.

Scheibenförmige Körper, das sind Zylinder (Achsenrichtung sei x_3) mit geringer Höhe h, die nur an dem Mantel des Zylinders (dem Rand der Scheibe) durch Kräfte und Verschiebungen in der x_1, x_2-Richtung beansprucht werden ($f_3 = 0$) — kurz *Scheibe*[2] genannt — sind für praktische Anwendungen wichtig; hier gelten also außer den Differentialgleichungen (1) bis (4) in dem Bereich

$$\{x_1, x_2\} \in \mathfrak{G}, \qquad |x_3| \leqq \tfrac{1}{2} h$$

die Randbedingungen

$$(5^*) \qquad \begin{aligned} \text{a)} \quad \sigma_{i3} &= 0 \quad \text{für} \quad x_3 = \pm \tfrac{1}{2} h; \quad \{x_1, x_2\} \in \mathfrak{G} \\ n_i \sigma_{ik} &= f_k \quad \text{für} \quad \{x_1, x_2\} \in \mathfrak{R}_1, \quad |x_3| \leqq \tfrac{1}{2} h \quad \text{mit} \quad f_3 \equiv 0. \\ \text{b)} \quad u_i &= u_i^* \quad \text{für} \quad \{x_1, x_2\} \in \mathfrak{R}_2; \quad |x_3| \leqq \tfrac{1}{2} h. \end{aligned}$$

Dabei bedeuten $\mathfrak{R}_1$, $\mathfrak{R}_2$ eine Zerlegung des Randes des zweidimensionalen Bereiches $\mathfrak{G}$. Der Einfachheit halber sollen dabei die in den Randbedingungen auftretenden Funktionen f_k und u_i^* als unabhängig von x_3 angenommen werden.

Für technische Zwecke gewinnt man nun zweidimensionale Gleichungen zur Lösung dieser Randwertaufgabe; dazu setzt man $\sigma_{i3} \equiv 0$, läßt die Gleichungen (3) mit $i = 3$ weg und nimmt alle Funktionen als nur von x_1 und x_2 abhängig an. Es entsteht dann das Gleichungssystem

$$(6) \qquad v_i = \text{Verschiebung}, \qquad \text{Verzerrung } \zeta_{ik} = \tfrac{1}{2} (v_{i|k} + v_{k|i}),$$

$$(7\text{a}) \qquad \zeta_{ik} = \frac{1}{2G} \tau_{ik} - \delta_{ik} \frac{\sum \tau_{ll}}{Em}$$

bzw. die Umkehrung

$$(7\text{b}) \qquad \tau_{ik} = 2G \zeta_{ik} + \frac{2G}{m-1} \delta_{ik} \left(\sum \zeta_{ll}\right)$$

und

$$(8) \qquad \tau_{ik|k} = 0,$$

was durch Elimination auf

$$(9) \qquad G \left\{ v_{i|kk} + \frac{m+1}{m-1} v_{k|ik} \right\} = 0$$

führt. Diese Differentialgleichungen (9), die Gleichungen der „zweidimensionalen Elastizitätstheorie" gelten im Innern des zweidimensionalen Bereiches $\mathfrak{G}$; dazu kommen noch Randbedingungen:

(10)
a) $n_i \tau_{ik} = f_k$ an dem Teil $\mathfrak{R}_1$ der Randkurve von $\mathfrak{G}$ wo die Randkräfte f_k gegeben sind ($k = 1, 2$),
b) $v_i = u_i^*$ an dem komplementären Teil $\mathfrak{R}_2$ des Randes, wo die Verschiebungen gegeben sind.

---

[2] Im Gegensatz zur technischen Platte, die auch — oder nur — seitliche Kräfte (in der x_3-Richtung) aufnehmen soll.

Es ist das Ziel dieser Untersuchung, diese durch Plausibilitätsbetrachtung (die insbesondere die Wahl der vernachlässigten Gleichung betrifft) gefundenen Gleichungen als wesentlich für dünne Scheiben nachzuweisen. Es wird dazu gezeigt werden, daß bei $h \to 0$ die über die Scheibendicke gemittelten Werte der Lösungen des Randwertproblems (4) (5) gegen die Lösung von (9) (10) streben.

II. Formulierung als Extremwertaufgabe

Für die dreidimensionale Elastizitätstheorie ist als Prinzip vom Minimum der Formänderungsarbeit bekannt, daß die Verschiebung durch die Extremumseigenschaft

$$(11)\quad A_3\{u\} \equiv G\iiint\left[\sum_{i,k}\gamma_{ik}^2+\frac{1}{m-2}\left(\sum\gamma_{ll}\right)^2\right]dx_1\,dx_2\,dx_3-\iint_{\mathfrak{R}_1}u_i f_i\,do=\text{Minimum}$$

$$(do=\text{Oberflächenelement})$$

charakterisiert werden kann; dabei werden alle hinreichend oft differenzierbaren u-Felder, die den geometrischen Bindungen (5*) genügen, zugelassen. Der Minimalwert bei der Scheibendicke h sei $a_3(h)$.

Analog gehört zu den Gleichungen (9) (10) das Extremwertproblem

$$(12)\quad A_2\{v\} \equiv G\iint\left[\sum_{i,k}\zeta_{ik}^2+\frac{1}{m-1}\left(\sum\zeta_{ll}\right)^2\right]dx_1\,dx_2-\int_{\mathfrak{R}_1}v_i f_i\,ds=\text{Minimum}$$

mit dem Minimalwert a_2.

Diese Charakterisierung der Lösungen durch Extremwerteigenschaften soll nun nach einem früheren Vorbild[3], welches beispielsweise die erste Randwertaufgabe bei $h\Delta\Delta u-\Delta u=0$ betraf, für den Beweis der behaupteten Limeseigenschaft benutzt werden[4].

III. Aufstellung zweier Ungleichungen

In A_2 werde

$$(13)\qquad v_i=\frac{1}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}u_i\,dx_3$$

als Vergleichsfunktion eingesetzt; dann folgt

$$\zeta_{ik}=\frac{1}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\gamma_{ik}\,dx_3$$

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[3] Morgenstern, D.: Singuläre Störungstheorie partieller Differentialgleichungen. J. Rat. Mechanics and Analysis **5**, 203—216 (1956).

[4] Die Existenz von Lösungen hat auf Grund der zugehörigen Extremwerteigenschaften K. Friedrichs nachgewiesen: On the boundary value problems of the theory of elasticity and Korn's inequality. Annals of Mathematics **48**, 441—471 (1947).

und aus der Schwarzschen Ungleichung erhält man

$$\zeta_{ik}^2 \leqq \frac{1}{h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \gamma_{ik}^2 \, d x_3 \quad \text{und} \quad (\sum \zeta_{ll})^2 \leqq \frac{1}{h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (\sum \gamma_{ll})^2 \, d x_3, \tag{14}$$

und es ergibt sich für

$$A_2\{v\} \leqq \frac{1}{h} \left\{ G \iiint \left[\sum_{i,k=1}^{2} \gamma_{ik}^2 + \frac{1}{m-1} \left(\sum_{l=1}^{2} \gamma_{ll} \right)^2 \right] d x_1 \, d x_2 \, d x_3 - \iint u_i f_i \, d o \right\}. \tag{15}$$

Nun gilt offensichtlich

$$\frac{1}{m-1} a^2 \leqq W^2 + \frac{1}{m-2} (a + W)^2 \tag{16}$$

und es bleibt also die folgende Ungleichung

$$\begin{aligned} A_2\{v\} \leqq \frac{1}{h} \Bigg\{ G \iiint \Bigg[\sum_{i,k=1}^{2} \gamma_{ik}^2 + \gamma_{33}^2 + \frac{1}{m-2} \left(\sum_{l=1}^{2} \gamma_{ll} + \gamma_{33} \right)^2 \Bigg] \times \\ \times d x_1 \, d x_2 \, d x_3 - \iint f_i \, u_i \, d o \Bigg\} \leqq \frac{1}{h} A_3\{u\}. \end{aligned} \tag{17}$$

Umgekehrt soll nun aus jedem Funktionenpaar v_1, v_2 ein u_i-Funktionensystem erzeugt und in den Ausdruck für A_3 eingesetzt werden:

$$u_i = v_i \; (i = 1, 2), \qquad u_3 = x_3 w(x_1, x_2); \tag{18}$$

dann gilt

$$\gamma_{ik} = \zeta_{ik} (i, k = 1, 2), \qquad \gamma_{i3} = \tfrac{1}{2} x_3 w_{|i} \; (i = 1, 2), \qquad \gamma_{33} = w,$$

und man erkennt, daß man nach der im Sinne der Ungleichung (16) optimalen Substitution

$$w = -\frac{1}{m-1} \sum_{l=1}^{2} \zeta_{ll}, \tag{19}$$

da die Integration in der x_3-Richtung leicht durchführbar ist, die Ungleichung

$$\begin{aligned} \frac{1}{h} A_3\{u\} \leqq G \iint \Bigg[\sum_{i,k=1}^{2} \zeta_{ik}^2 + \frac{1}{m-1} \left(\sum_{l=1}^{2} \zeta_{ll} \right)^2 + \frac{1}{2} \frac{h^2}{12} (\operatorname{grad} w)^2 \Bigg] \times \\ \times d x_1 \, d x_2 - \int_{\Re_1} v_i f_i \, d s = A_2\{v\} + G \frac{h^2}{24} \iint (\operatorname{grad} w)^2 \, d x_1 \, d x_2 \end{aligned} \tag{20}$$

erhält.

IV. Abschluß des Beweises

Zunächst erkennt man aus der Ungleichung (17), daß für die aus der Lösung $u^{(h)}$ für die Scheibe der Dicke h durch (13) gewonnenen Funktionen $v^{(h)}$ gilt

$$A_2\{v^{(h)}\} \leqq \frac{1}{h} A_3\{u^{(h)}\} = \frac{1}{h} a_3(h). \tag{21}$$

Andrerseits gilt wegen (20) für das aus der Lösung des zweidimensionalen Problems v gewonnenen u

$$\begin{aligned}\frac{1}{h}a_3(h) = \frac{1}{h}\min A_3 \leqq \frac{1}{h}A_3\{u\} \leqq A_2\{v\} + G\frac{h^2}{24}\iint(\operatorname{grad} w)^2\,dx_1\,dx_2\\ = a_2 + G\frac{h^2}{24}\iint(\operatorname{grad} w)^2\,dx_1\,dx_2,\end{aligned} \tag{22}$$

wobei $w = -\frac{1}{m-1}\sum_{l=1}^{2}\zeta_{ll}$ einzusetzen ist.

Wenn also dies letzte Integral konvergiert, erkennt man

$$\overline{\lim_{h\to 0}}\,\frac{1}{h}a_3(h) \leqq a_2 \tag{23}$$

und wegen $A_2\{v^{(h)}\} \geqq a_2$ und (21) auch

$$\lim_{h\to 0} A_2\{v^{(h)}\} = a_2. \tag{24}$$

Falls für die Lösung des zweidimensionalen Problems das Integral $\iint(\operatorname{grad} w)^2 dx_1 dx_2$ divergiert, kann man statt dessen eine Näherungsfunktion verwenden, für die das Integral endlich bleibt, aber a_2 bis auf eine beliebig kleine Differenz erreicht wird. Es folgt dann dasselbe Ergebnis.

In dem Hilbertschen Raum der Funktionenpaare v_1, v_2 mit der Dirichlet-Norm

$$\|v^2\| \equiv (v, v) = G\iint\Big[\sum\zeta_{ik}^2 + \frac{1}{m-1}(\sum\zeta_{ll})^2\Big]\,dx_1\,dx_2 \tag{25}$$

läßt sich A_2 so ausdrücken

$$A_2\{v\} = (v, v) - 2(v, g) \equiv \|v-g\|^2 - \|g\|^2, \tag{26}$$

wobei g zu der Lösung des zweidimensionalen Problems gehört. Für die Abstände von diesem Element g gilt dann, wenn $\mathfrak{M}$ die lineare Menge der erlaubten Funktionenpaare bezeichnet, die die Randbedingungen (10) erfüllen,

$$\inf_{s\in\mathfrak{M}}\|s-g\|^2 = a_2 + \|g\|^2 = \|v-g\|^2 \tag{27}$$

und

$$\|v^{(h)} - g\|^2 \to a_2 + \|g\|^2 \qquad (v^{(h)}\in\mathfrak{M}), \tag{28}$$

woraus nach dem Lemma vom eindeutig bestimmten Lot auf konvexe Mengen die Konvergenz $v^{(h)}\to v$ im Sinne unserer Norm (25) folgt.

Damit ist also bewiesen, daß die nach (13) gemittelten Werte der Lösungen des dreidimensionalen Problems bei $h\to 0$ im Mittel gegen die Lösung der Gleichungen (9) (10) konvergieren[5]. Aus dieser Mittelkonvergenz im Sinne der Norm (25) folgt, wie leicht gezeigt werden kann, auch die Mittelkonvergenz im üblichen Quadratmittel, wenn man v_1, v_2 nur bis auf Transformationen $v_1\to v_1 + a x_2 + b$, $v_2\to v_2 - a x_1 + c$, die infinitesimalen Bewegungen entsprechen, als bestimmt ansieht.

---

[5] Diese zweidimensionalen Gleichungen sind nicht zu verwechseln mit den Gln. (4) für x_3-unabhängige Spannungszustände (unendlich dicke Scheiben), wo alle Größen von x_3 unabhängig sind.

V. Schlußbemerkung

Außer der sich hier anschließenden Frage nach der punktweisen Konvergenz, die im Anschluß an dieses Ergebnis zu lösen möglich sein dürfte, gibt es eine Reihe anderer begrifflicher Lücken beim Aufbau der Mechanik, insbesondere der in der ,,technischen Mechanik" verwendeten Näherungstheorien, die durch ähnliche mathematische Grenzwertsätze geschlossen werden sollten; so zum Beispiel fehlt eine exakte Begründung für die technische Näherungstheorie der Balkenlehre.

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